6. Functions and equations

Functions and their graphs

You should be able to:

- use the features of your calculator to graph a variety of functions
- use your calculator to investigate key features of graphs, including maximum values, minimum values, points of intersections, and zeros
- find a value of a function
- find the range of a function on a given domain
- identify horizontal and vertical asymptotes in functions where appropriate
- solve equations using a graphical approach.

You should know:

- a function is usually denoted by a small letter of the alphabet, such as f or g; the notation $f(x)$ or $g(x)$ is the value of the function at x
- the domain of a function is the set of x-values for which a function is defined
- the range of a function is best found by looking at the maximum and minimum values of the graph of the function on a given domain
- the line $x = a$ is a vertical asymptote if $f(a)$ is undefined and $f(x) \to \pm\infty$ as $x \to a$
- horizontal asymptotes tell us how the graph of a function behaves when x gets very large positively or negatively. They are often found in rational and exponential functions.

Example

Let $g(x) = x^4 - 2x^3 + x^2 - 2$.

(a) Write down the values of x for which $g(x) = 0$.

Use your GDC to graph $g(x)$ and then use the "zero" or "root" function to find the solutions to the equation.

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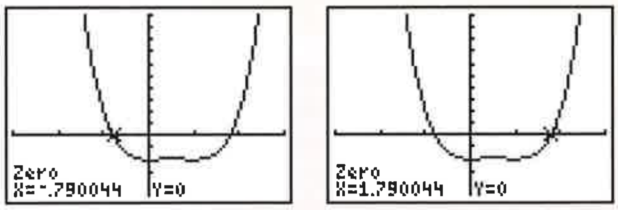

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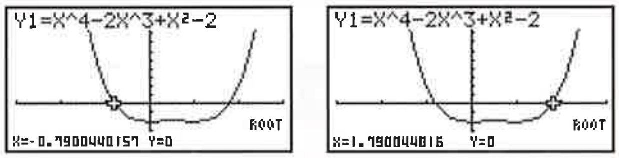
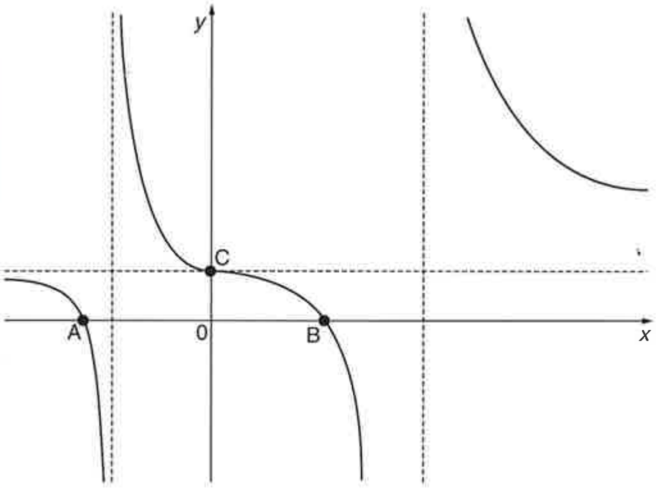


$x = -0.790$ and $x = 1.79$

Functions and their graphs (continued)

Let $f(x) = \frac{2x^3}{g(x)} + 1$. The graph has vertical asymptotes at $x = a$ and $x = b$ with $a < b$. Part of the graph of f is shown below.



(b) (i) Write down the value of a and of b.

As $g(x)$ is in the denominator of the function, a and b are the values of x for which $g(x) = 0$. These values were found in part (a).

$a = -0.790$ and $b = 1.79$ as $a < b$.

(ii) Find the coordinates of C.

The point C is the y-intercept. We need to find $f(0)$.

$$f(0) = \frac{2(0)}{g(0)} + 1 = 1$$

Therefore, C (0, 1)

(c) The graph has a horizontal asymptote at $y = 1$. Explain why the value of $f(x)$ approaches 1 as x gets very large.

The degree of $g(x)$ is greater than the degree of the numerator. Therefore, when x gets very large, the term $\frac{2x^3}{g(x)}$ approaches zero and $f(x)$ will approach the value of 1.

Be prepared

- Know the major features of your GDC.
- When asked to sketch a graph, make effective use of the zoom and trace features to ensure that you are looking at an accurate representation of the graph. Set your window equal to the given domain.
- When finding numerical values such as maximum and minimum points, use the appropriate menus and not the trace features of your GDC.

Composite and inverse functions

You should be able to:

- find the composition of two functions
- find the value of the composition of two functions for a particular value of x
- use the graph of a function to decide whether the function has an inverse
- sketch the graph of the inverse of a function from the graph of a function
- verify that two functions are inverses of each other.

You should know:

- the composite function $f \circ g$ can also be denoted by $(f \circ g)(x)$ or more simply, $f(g(x))$
- the composition $f \circ g$ is the operation of applying g then f; the composition $g \circ f$ is the operation of applying f then g
- if f is a function that maps x to $f(x)$, then the inverse function f^{-1} maps $f(x)$ back onto x
- geometrically, the graph of the inverse function is a reflection of the original function in the line $y = x$
- the domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}
- to find the inverse of a function analytically, simply switch the positions of x and y and then solve the resulting equation for y
- a function $g(x)$ is the inverse of $f(x)$ if and only if $f(g(x)) = g(f(x)) = x$.

Example

Consider the functions $f(x) = 2x$ and $g(x) = \frac{1}{x-3}$, $x \neq 3$.

(a) Find $(f \circ g)(4)$.

Since $(f \circ g)(4) = f(g(4))$ then we evaluate $g(4)$ first, then substitute the result into the function f.

So $g(4) = 1$ and $f(1) = 2$, therefore, $(f \circ g)(4) = 2$.

(b) Find $g^{-1}(x)$.

Switching the positions of x and y, we have $x = \frac{1}{y-3}$.

Solving this result for y gives $y = \frac{1}{x} + 3$.

Hence, $g^{-1}(x) = \frac{1}{x} + 3$.

(c) Write down the domain of g^{-1}.

Note that $g^{-1}(x)$ is undefined when $x = 0$, so the domain of g^{-1} includes all real numbers except 0.

Be prepared

- The order in which the functions are to be composed is extremely important.
- Proceed with caution when simplifying expressions resulting from a composition.

Quadratic functions

You should be able to:

- move confidently between the three different forms of a quadratic function
- extract important information about the graph of a quadratic function from its equation
- solve quadratic equations using a variety of techniques including factorization, the quadratic formula or the GDC
- write equations of quadratic functions given information about their graphs
- use the discriminant (Δ) to determine the nature of the roots of a quadratic function.

You should know:

- the vertex of the quadratic function $f(x) = a(x - h)^2 + k$ is at (h, k)
- the roots of the quadratic function $f(x) = a(x - b)(x - c)$ occur at $x = b$ and $x = c$
- the y-intercept of the function $f(x) = ax^2 + bx + c$ has coordinates $(0, c)$
- the equation of the axis of symmetry is $x = -\dfrac{b}{2a}$
- the graph of a quadratic function opens up when the leading coefficient, a, is positive and opens down when a is negative
- the vertex of a parabola can be used to solve optimization problems that are quadratic
- a quadratic function has two real roots when $\Delta > 0$, one real (double) root when $\Delta = 0$ and no real roots when $\Delta < 0$.

Example

Let $f(x) = a(x - 4)^2 + 8$.

(a) Write down the coordinates of the vertex of the curve of f.

Since the function has the $f(x) = a(x - h)^2 + k$ form, the vertex has coordinates (h, k).

Therefore, the vertex has coordinates $(4, 8)$.

(b) Given that $f(7) = -10$, find the value of a.

Substituting into the equation, we have:

$-10 = a(7 - 4)^2 + 8$

$a = -2$

(c) Find the y-intercept of the curve of f.

The y-intercept occurs when $x = 0$.

Therefore, $y = -2(0 - 4)^2 + 8$

$y = -24$

Alternatively, we could have rewritten $f(x)$ in the form $y = ax^2 + bx + c$.

Be prepared

- Your calculator is an efficient tool for quickly finding the vertex of a quadratic function and then expressing $ax^2 + bx + c$ in the form $a(x - h)^2 + k$.
- It may be necessary to reject one of the solutions of a quadratic equation if it makes no sense in the context of the question.

Transformations of graphs

You should be able to:

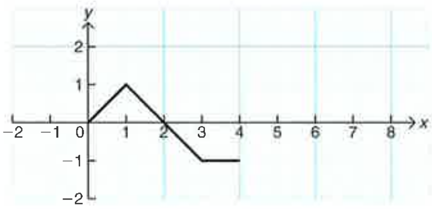
- identify and describe the following transformations: translations, reflections and stretches
- find the equation of the image function following one or more transformations
- sketch the image of a function under a transformation
- give a full geometric description of the transformation(s) that map a function or its graph onto its image.

You should know:

- a translation is described by a vector $\begin{pmatrix} p \\ q \end{pmatrix}$, which shifts a graph horizontally by p units and vertically by q units without changing the shape of the graph
- a function of the form $y = f(x - p) + q$ translates the graph of f by the vector $\begin{pmatrix} p \\ q \end{pmatrix}$. All points (x, y) are mapped onto $(x + p, y + q)$
- a function of the form $y = af(x)$ represents a vertical stretch by a scale factor of a. When a is greater than 1 or less than −1 the graph moves away from the x-axis and it moves towards the x-axis when a is between −1 and 1; all points (x, y) are mapped onto (x, ay) and the shape of the graph is changed
- a function of the form $y = f(kx)$ represents a horizontal stretch by a scale factor of $\frac{1}{k}$. The graph moves away from the y-axis when k is a value between −1 and 1 and it moves towards the y-axis when k is greater than 1 or less than −1; all points (x, y) are mapped onto $\left(\frac{x}{k}, y\right)$ and the shape of the graph is changed
- a function of the form $y = -f(x)$ represents a reflection in the x-axis such that all points (x, y) are mapped onto $(x, -y)$; the shape of the graph is unchanged
- a function of the form $y = f(-x)$ represents a reflection in the y-axis such that all points (x, y) are mapped onto $(-x, y)$; the shape of the graph is unchanged.

Example

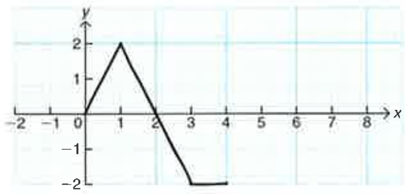
The graph of $y = f(x)$ is shown below.



On each of the following diagrams, draw the required graph.

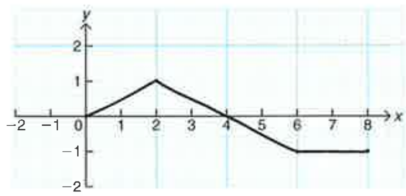
(a) $y = 2f(x)$

This function will stretch the graph of f vertically away from the x-axis by a factor of 2. As such, all points (x, y) will be mapped onto $(x, 2y)$.



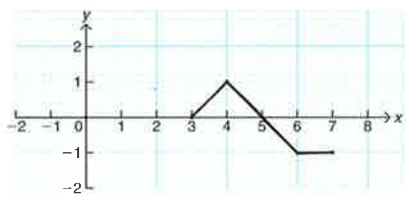
(b) $y = f\left(\frac{x}{2}\right)$

This function will stretch the graph of f horizontally away from the y-axis by a factor of 2. As such, all points (x, y) will be mapped onto $(2x, y)$.



(c) $y = f(x - 3)$

This function will translate the graph of f horizontally to the right by 3 units. As such, all points (x, y) will be mapped onto $(x + 3, y)$. Notice that the shape of the graph does not change.



(d) The point A(3, −1) is on the graph of f. The point A′ is the corresponding point on the graph of $y = -f(x + 1) - 2$. Find the coordinates of A′.

This function will reflect the graph of f in the x-axis, translate it to the left by 1 unit and down by 2 units. As such, all points (x, y) will be mapped onto $(x - 1, -y - 2)$.

Therefore, $(3, -1) \longrightarrow (3 - 1, -(-1) - 2) \longrightarrow (2, -1)$

So A′ has coordinates $(2, -1)$.

Be prepared

- Horizontal transformations such as stretches can be tricky. Remember that a horizontal stretch by a factor of $k > 1$ "stretches" the graph towards the y-axis.

Exponential functions

You should be able to:

- investigate functions of the form $f(x) = a^x$ and interpret the effect on the graph of changing the base
- use a GDC to solve exponential equations and inequalities
- use exponential functions to solve problems involving growth and decay.

You should know:

- the function $f(x) = a^x$ is an exponential function with base a, which can be any positive real number except 1
- if $a > 1$, the graph depicts exponential growth, and if $0 < a < 1$, the graph demonstrates exponential decay
- the graphs of all functions $f(x) = a^x$ pass through the point (0, 1) and have a horizontal asymptote at $y = 0$
- the function $f(x) = a^x + b$ has a horizontal asymptote at $f(x) = b$
- the function $f(x) = e^x$ is an exponential function with base e
- functions of the form $f(x) = a^x$ can be expressed with base e as $f(x) = e^{x \ln a}$
- the terms of a geometric sequence, if plotted, will illustrate exponential growth if the common ratio $r > 1$ and will illustrate exponential decay if $0 < r < 1$
- exponential growth and decay are modelled by the function $f(t) = A_0 e^{rt}$, where A_0 is the initial condition and r is the rate of growth over period of time t. If $r < 0$, exponential decay is being modelled.

Example

A group of 10 leopards is introduced to a game park. After t years, the number of leopards, N, is modelled by $N(t) = 10e^{0.4t}$.

(a) How many leopards are there after 2 years?

This problem is asking us to find $N(2)$. Substituting $t = 2$ into the function,

$N(2) = 10e^{0.4(2)}$

$= 22.3$

The number of leopards is best reported as 22.

The answer is easily obtained from the graph of N:

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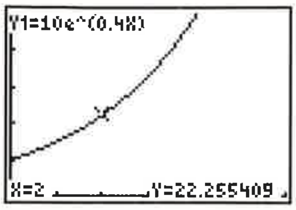

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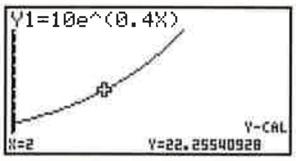

(b) How long will it take for the number of leopards to reach 100?

We are being asked to find t when $N(t) = 100$. We can either take an analytical approach involving logarithms or a geometric approach using the GDC.

Analytical approach

$10e^{0.4t} = 100 \Rightarrow e^{0.4t} = 10$

Taking the natural logarithm of both sides,

$\ln e^{0.4t} = \ln 10 \Rightarrow 0.4t = \ln 10$

$t = 5.76$

So in approximately 5 years and 9 months, the population of leopards will reach 100.

Graphical approach

Graph the function $N(t)$ and find where it intersects the graph of $y = 100$.

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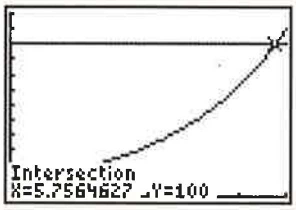

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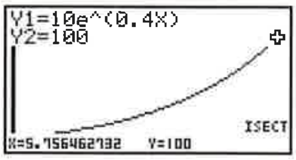


The intersection occurs when $x \approx 5.76$ and we reach the same conclusion as above.

Be prepared

- Write an inequality when words such as "exceeds" or "not more than" are used in a problem. Then be sure to interpret your answer appropriately in the context of the question.
- Find the derivative of an exponential function when asked about the **rate** of growth or decay.
- Use logarithms or a GDC to solve problems that require you to find the value of the exponent, often interpreted as time.

Logarithmic functions

You should be able to:

- sketch the graphs of logarithmic functions for different bases
- describe the effect on the graph of changing the base of a logarithmic function
- state the characteristics of the graphs of the natural logarithmic function.

You should know:

- the logarithmic function is the inverse of the exponential function
- the function $f(x) = \log_a x$ is a logarithmic function with base a, which can be any positive real number except 1
- if $a > 1$, the graph increases over its entire domain, and if $0 < a < 1$, it decreases
- the function f has the y-axis as a vertical asymptote; a logarithmic function of the form $y = \log_a(bx + c)$ has a vertical asymptote at $y = -\frac{c}{b}$
- the function $g(x) = \ln x$ is a logarithmic function with base e
- the graphs of the logarithmic functions f and g always pass through the point (1, 0)
- logarithmic functions can be used to model Richter scales, the magnitude of sound, and the pH of a solution.

Example

Let $f(x) = \log_2(x - 1)$, $x > 1$.

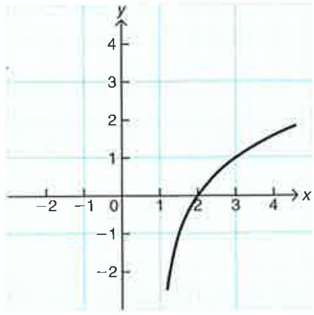
(a) Write down the value of $f(3)$.

Substituting $x = 3$ into $f(x)$, we are then required to find the value of $\log_2 2$. The base 2 must be raised to the first power to give 2. That is, if $\log_2 2 = x$, then $2^x = 2^1$ and $x = 1$. Hence, $\log_2 2 = 1$.

(b) Write down the equation of the vertical asymptote.

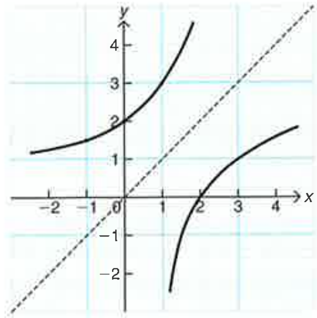
The graph of f is the graph of $\log_2 x$ translated 1 unit to the right. So the vertical asymptote is found at $x = 1$.

(c) The diagram below shows part of the graph of f.



On the same diagram, sketch the graph of f^{-1}.

The graph of f^{-1} is a reflection of the graph of f in the line $y = x$. Therefore, all points (x, y) on the graph of f are mapped onto all points (y, x) on the graph of f^{-1}.



Be prepared

- Be prepared not to trust your calculator when graphing logarithmic functions. The graph below shows the function $y = \log_2(x + 2)$. It would appear that there is an end point on the left but we know this is not the case. This is one limitation of your GDC.

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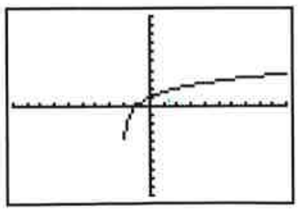

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