9. Vectors

Vector algebra and geometry

You should be able to:

- represent vectors as column vectors, or using base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, for example $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$
- find the distance between points in the plane and in three dimensions
- use both geometric and algebraic approaches for the following:
  - sum and difference of two vectors
  - multiplication of a vector by a scalar, $k\boldsymbol{v}$
  - magnitude of a vector, $|\boldsymbol{v}|$
  - the vector $-\boldsymbol{v}$
  - the zero vector, $\boldsymbol{v} + (-\boldsymbol{v}) = 0$
- express a vector $\overrightarrow{AB}$ in terms of its position vectors as $\overrightarrow{OB} - \overrightarrow{OA}$
- find a unit vector parallel to a given vector.

You should know:

- a vector shows both **direction** and **magnitude**
- vectors are displacements in the plane and in three dimensions
- the magnitude of a vector is denoted by $|\boldsymbol{v}|$ and is determined by $|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- a vector $-\boldsymbol{v}$ has the same magnitude as $\boldsymbol{v}$ but is in the opposite direction
- the vector $\overrightarrow{AB}$ is the vector that begins at A and ends at B
- a position vector $\overrightarrow{OA}$ begins at the origin and ends at A
- a vector of magnitude 1 is called a **unit** vector
- $k\boldsymbol{v}$ is a vector of magnitude $k|\boldsymbol{v}|$ in the same direction as $\boldsymbol{v}$
- the sum or difference of two vectors can be found by adding or subtracting their corresponding components
- the base vectors $\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ have a length of 1 unit and are parallel to the coordinate axes
- it is sometimes useful to find a vector of a given length. To do this, multiply a unit vector by whatever length is desired. For example, a vector of length 10 in the same direction as $\boldsymbol{v} = \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$ can be found by multiplying the unit vector of $\boldsymbol{v}$ by 10, that is $10 \times \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$.

Example

Three of the four vertices of a quadrilateral are given as A(2, 3, 1), B(6, 5, 4) and C(3, 1, 5).

(a) Find the vector $\overrightarrow{AB}$, expressing your answer in the form $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$.

The vector $\overrightarrow{AB}$ is found by subtracting the position vector of A from that of B. That is, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. Therefore,

$$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

So in the required form, $\overrightarrow{AB} = 4\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.

(b) Find $|\overrightarrow{AB}|$.

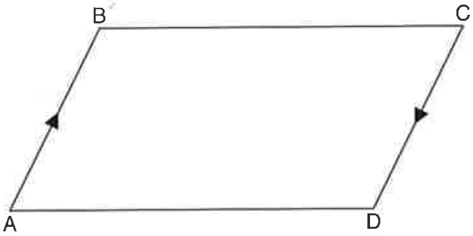
The magnitude of $\overrightarrow{AB}$ can be found using the formula for the magnitude of a vector.

$$|\overrightarrow{AB}| = \sqrt{4^2 + 2^2 + 3^2} = \sqrt{29}$$

Vector algebra and geometry (continued)

(c) Given that the quadrilateral is a parallelogram, find the position vector of the fourth vertex, D.

The diagram shows the parallelogram ABCD with $\overrightarrow{AB}$ parallel to $\overrightarrow{DC}$. To find the coordinates of D, we add the vector $\overrightarrow{CD}$ to the position vector $\overrightarrow{OC}$. We also see from the diagram that $\overrightarrow{CD} = -\overrightarrow{AB}$.



Therefore,

$\overrightarrow{CD} = -\overrightarrow{AB} = -4i - 2j - 3k$, so

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

Hence, D has coordinates (−1, −1, 2) or position vector $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$.

Be prepared

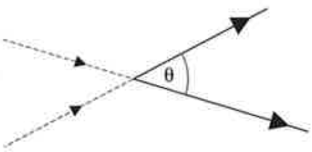
- You need to be able to solve vector problems set in either two or three dimensions.

Scalar product

You should be able to:

- find the scalar product of two vectors
- find the angle between two vectors
- state with reasons whether two vectors are parallel, perpendicular or neither.

You should know:

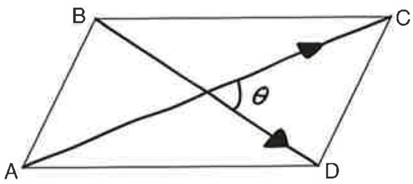


- the angle θ between vectors is the angle formed when the starting points or ending points of two vectors coincide
- the scalar product is defined by $\boldsymbol{v} \cdot \boldsymbol{w} = |\boldsymbol{v}||\boldsymbol{w}| \cos\theta$, where θ is the angle between the two vectors
- if $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, then the component formula for the scalar product is given by $\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$
- if the scalar product is less than zero, that is $\boldsymbol{v} \cdot \boldsymbol{w} < 0$, the angle between the two vectors is obtuse, and if $\boldsymbol{v} \cdot \boldsymbol{w} > 0$, the angle between the vectors is acute
- vectors are perpendicular if and only if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$
- vectors are parallel if $\boldsymbol{v} \cdot \boldsymbol{w} = \pm|\boldsymbol{v}||\boldsymbol{w}|$ as the angle between them is either 0 or π.

Example

The vertices of a parallelogram are given as A(2, 3, 1), B(6, 5, 4), C(3, 1, 5) and D(−1, −1, 2). Find the acute angle between the diagonals of the parallelogram.

Sketching a diagram will help you see what vectors to use. The two diagonals are AC and BD.



To find the correct angle θ, we require the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$.

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ -6 \\ -2 \end{pmatrix}$$

Using the formula $\cos\theta = \dfrac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|\overrightarrow{AC}||\overrightarrow{BD}|}$, *we have*

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = (1)(-7) + (-2)(-6) + (4)(-2) = -3$$

$$|\overrightarrow{AC}| = \sqrt{21}$$

$$|\overrightarrow{BD}| = \sqrt{89}$$

So $\cos\theta = \dfrac{-3}{\sqrt{21} \times \sqrt{89}}$ *and θ = 1.64 radians correct to three significant figures.*

Be prepared

- State clearly which vectors you are using when finding the angle between two vectors. Remember that $\overrightarrow{AB}$ is not the same as $\overrightarrow{BA}$.
- Show working or evidence of finding the scalar product and the magnitudes of each of the two vectors.

Vector equation of a line

You should be able to:

- express the equation of a line in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{b}$ is a vector representing the direction of the line
- interpret t as time, $\boldsymbol{b}$ as the velocity and $|\boldsymbol{b}|$ as the speed of an object in motion.

You should know:

- you can write the vector equation of a line if you know the coordinates of any two points A and B on the line. The vector $\overrightarrow{AB}$ gives the direction of the line and either $\overrightarrow{OA}$ or $\overrightarrow{OB}$ can be used as the position vector of a point on the line
- the vector equation of a line is not unique
- an object with a velocity vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ moves parallel to the vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with a speed of $\sqrt{u_1^2 + u_2^2}$.

Example

A line L passes through the points A(3, 2, 1) and B(1, 5, 3).

(a) Find the vector $\overrightarrow{AB}$.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$

(b) Write down a vector equation of the line L in the form $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$.

Using $\overrightarrow{AB}$ as the direction of the line and the position vector $\overrightarrow{OA}$ of point A on the line, we have $r = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$.

Note that we could have used the position vector of point B on the line and/or any scalar multiple of $\overrightarrow{AB}$ to write a different equation representing the same line. Hence, the vector equation of a line is not unique.

Be prepared

- Do not confuse the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. They are not interchangeable.
- Write your answer as an equation.
- Motion problems can be set in either two or three dimensions.
- Giving an answer such as $L = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}$ does not give a correct equation for a line.

Intersections and angles

You should be able to:

- find the angle between two lines
- find the coordinates of the position vector of the point of intersection of two lines.

You should know:

- to find the angle between two lines expressed in vector form, find the angle between their direction vectors
- to find the position vector of the point of intersection of two lines expressed in vector form, you must set their x, y and z components equal to each other.

Example

A line L_1 has equation

$$\boldsymbol{r}_1 = (-5 + s)\boldsymbol{i} + (11 - 2s)\boldsymbol{j} + (-8 + 3s)\boldsymbol{k}.$$

A second line L_2 has equation

$$\boldsymbol{r}_2 = 2\boldsymbol{i} + 9\boldsymbol{j} + 13\boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}).$$

Find the angle between lines L_1 and L_2.

Although L_1 is expressed differently, it can be written in the form $r = a + sb$ as $r_1 = -5i + 11j - 8k + s(i - 2j + 3k)$.

The direction vector of L_1 is $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. The direction vector of L_2 is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. The angle θ between the lines is the same as the angle between the direction vectors. Therefore,

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}{\sqrt{14} \times \sqrt{14}} = 0.42857\ldots$$

and $\theta = 1.13$ radians to three significant figures.

Be prepared

- Use a different parameter, such as s and t, when expressing two different lines in vector form.