FUNCTIONS AND EQUATIONS

Basics of Functions

A function is an algebraic rule which shows how one set of numbers is related to, or obtained from another set. Functions often model real-life situations, so it is necessary to understand the notation used and the different types of function which may be used.

Defining functions: A function is defined using the notation f: $x \to \ldots$ For example, f: $x \to x^2 - 1$. An alternative notation is $f(x) = x^2 - 1$ so that, for example, $f(3) = 3^2 - 1 = 8$. The x value put in to the function is called the *object* and the value of the function which results is called the *image*.

Read the definition as: "The function f takes any number x and turns it into $x^2 - 1$"

Domain: The set of values to be input to a function is called the *domain* of the function. In many functions, *any* value can be input, in which case the domain is $x \in \mathbb{R}$. However, the domain may be restricted for two reasons:

- Certain values of x may give impossible results, such as division by 0 or square root of a negative. For example, the function $f: x \to \frac{x}{x-4}$ has the domain restriction $x \neq 4$.
- The domain and such restrictions will always form part of the function definition.
- For the purposes of the question, the domain may be "artificially" restricted. In the following example, the only values of x which are to be input to the function are 3, 4 and 5. f: $x \to 2x - 3$, $(3 \le x \le 5,\ x \in \mathbb{Z})$

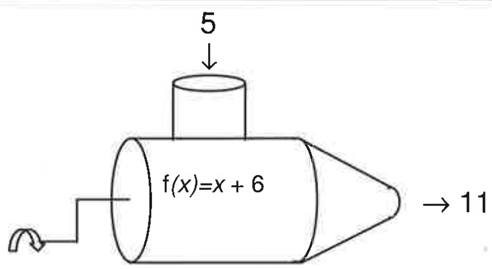
Note that a function can also be defined in words:

f: x $\to$ distance from nearest integer. What is f(2.8) and what is the range?

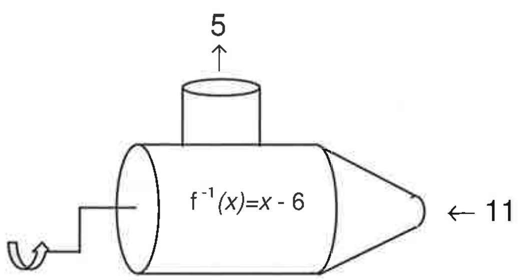
The nearest integer to 2.8 is 3, so f(2.8) = 0.2. *The distance can never be more than 0.5 so the range is* $0 \le x \le 0.5$

Range: The set of values produced by a function is called the *range*. In the last example above, the range would be $3 \le f(x) \le 7$. Generally, the easiest way to find the range of a function is to look at its graph: the range is the complete set of possible y values.

Imagine a "function machine." When the handle is turned, the 5 drops in the top, and the function machine turns it into an 11! This image is used in the next sections.



Inverse functions: An inverse function "reverses" the effect of a function. The inverse of add 2 is subtract 2. The inverse of squaring is square rooting. In terms of the function machine, just turn the handle the other way and the 11 turns back into a 5. The notation for an inverse function is $f^{-1}(x)$. A general method for finding inverse functions is given over the page.



To work out the inverse of a function – particularly a more complex one – the method is:

- Write the function in the form y = the function
- Replace the y with an x and all the x's with y's.
- Make y the subject – you will have the inverse function.

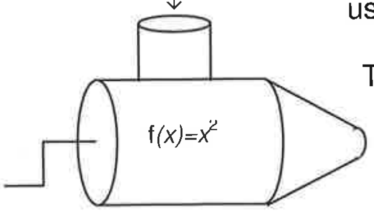
Another point to note about inverse functions is that the range of a function becomes the domain of its inverse.

Find the inverse function of f: $x \rightarrow \sqrt{(x+2)}$, $x \geq -2$. What is the domain of f^{-1}?
(Note the domain restriction which prevents square roots of negative numbers).

$$y = \sqrt{x+2}$$
$$x = \sqrt{y+2}$$
$$x^2 = y+2$$
$$y = x^2 - 2 \Rightarrow f^{-1}(x) = x^2 - 2$$

The range of the function is $f(x) \geq 0$. So the domain of the inverse function is $x \geq 0$.

Composite functions: If the image numbers from one function are input to another one, the result is a *composite function*. If $f(x) = x^2$ and $g(x) = x - 1$, then $f(g(3)) = f(2) = 4$. This is not the same as $g(f(3)) = g(9) = 8$. It is important to understand that the functions are not being multiplied together; a number is being put through one function, then the other. This can be illustrated using the function machines on the left.



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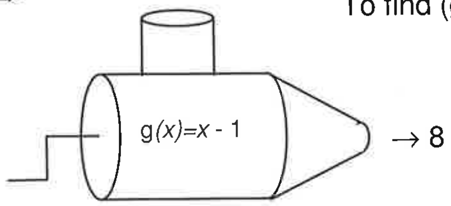


The actual notation used (to avoid multiple brackets) is $(g \circ f)(3)$. Say this as "g of f of 3" and remember that 3 is put into f first and then into g.

To find $(g \circ f)(x)$, work like this:

$g(f(x)) = g(x^2)$ Now function g in words is "subtract 1", so we end up with $x^2 - 1$. $(f \circ g)(x) = f(x - 1)$. Function f is "square" so we end up with $(x - 1)^2$.

YOU SOLVE

Given the functions $f(x) = x^2$ and $g(x) = \sin x$, find

a) An expression for $(g \circ f)(x)$

b) The exact value of $(f \circ g)(2\pi/3)$

sin x^2, 3/4

Functions and Graphs with a GDC

In this section of the syllabus, perhaps more than any other, you are expected to be able to use your graphic calculator for a wide range of techniques. You will use the calculator in four ways:

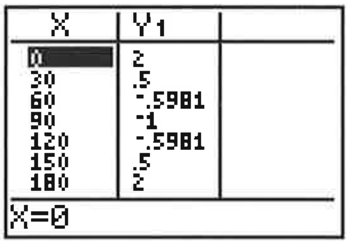
- As a simple "scientific" calculator (ie to do calculations)
- To check answers to questions you have worked out "by hand"
- To work out answers more quickly (especially for graphical questions)
- To answer questions which cannot be done in any other way

Functions: You should be able to use function keys with confidence. Make sure you know how to key in these functions:

Function	Examples
Squaring and other powers	3.2^2, 5.18^4, $(-3)^5$, -3^5
Square roots and other roots	$\sqrt{3.8}$, $\sqrt[4]{28}$
Trigonometric functions *(Make sure your calculator is set in degrees)*	sin33°, $\cos^{-1}0.867$

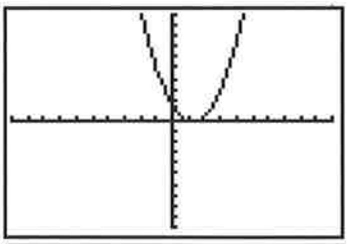
You also need to know how to use these keys to type in a function of *x*, eg: $y = \sqrt[3]{\dfrac{x}{x-1}}$

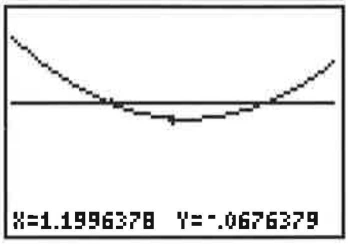
Tables: GDCs have a facility to work out a table of values for a function. Having input the function in the form $y = f(x)$ you can set up a table by selecting the first *x* value and then the steps by which you want *x* to increase. In this example, the function $y = 2 - 3\sin x$ has been entered into the function editor, and then a table created starting with $x = 0$ and increasing *x* in steps of 30. This can be helpful if you need to know several values, if you want to plot a graph by hand or if you're having difficulty creating the appropriate scales for a calculator plot – the table indicates the lowest and highest values of *y*.



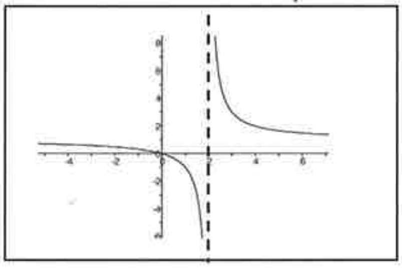
Drawing graphs: Three important points to remember when drawing and using GDC graphs.

- Make sure the function you type into the editor is actually the same as in the question. You may, for example, have to use brackets which aren't actually required on the written page. 2^{x+3}, if typed as 2 ^ x + 3, will work out values of $2^x + 3$. You need a bracket: 2 ^ (x + 3)
- The GDC has a few standard sets of scales, but you will probably have to set up the "window" yourself in order to see the required part of the graph. You may well have to zoom into a part of the graph to see exactly what is happening. The two screenshots on the right are of the same graph, but only the lower one shows the intersections with the *x*-axis.





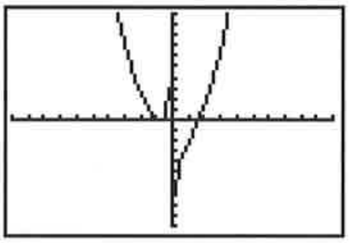
- The GDC can give you the values of key points such as intersections with the axes, points where lines intersect, turning points and so on. If you want to read off your own point, make sure you know the scales being used, ie how much each mark on the axes is worth.



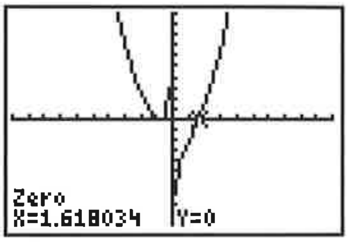
This graph also has a horizontal asymptote at $y = 1$. Note that some calculators draw vertical asymptotes in because they join all the points – but the asymptote is not part of the graph.

Vertical asymptotes: A graph such as $y = 2^x$ has a horizontal asymptote because as x gets smaller, the values of y get ever closer to 0 without ever reaching it. Some functions have graphs with vertical asymptotes which arise because division by 0 is impossible. For example, $y = \dfrac{x}{x-2}$ (left) will have a vertical asymptote at $x = 2$; as x gets closer to 2, the bottom line gets closer to 0.

Solving equations: GDCs have built in equation solvers. They can sometimes be a little cumbersome to use, so it is probably better to use graphs to solve equations. The easiest way to do this is to ensure your equation has a 0 on the right hand side because then all you have to do is find out where the graph cuts the axis.



For example, solve $x^2 - 2 = \dfrac{1}{x}$, $x > 0$. First we need to rewrite this equation as $x^2 - 2 - \dfrac{1}{x} = 0$. The graph is shown on the left.



Now use the "zero" or "root" feature to find where the graph cuts the x-axis and this will be the solution to the equation. $x = 1.618$

YOU SOLVE

$f(x) = x^3 \times 2^{-x}$, $x \geq 0$.

a) **Sketch the graph of $f(x)$, showing its asymptotic behaviour.**
Note the domain of the function.

b) **Find the co-ordinates of the maximum point, and hence state the range of $f(x)$.**
Once you know the y-coordinate of the maximum, you can use this to write down the range; ie the set of possible values of the function. Again, note the domain.

c) **Draw a line on your graph to show that $f(x) = 1$ has two solutions.**

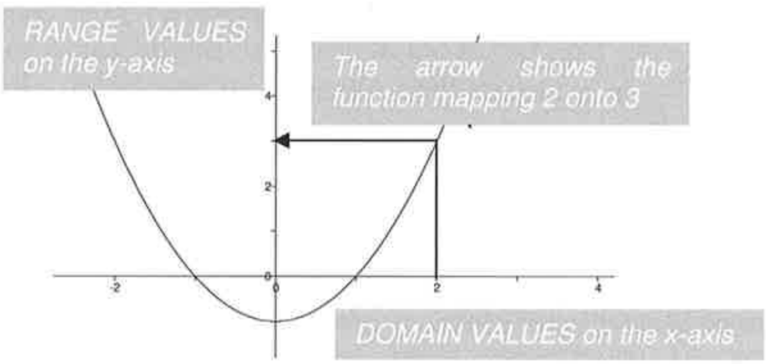
d) **Find the solutions to $f(x) = 1$, giving your answers to 3 significant figures.**
Either draw $y = 1$ on your calculator and find the two points of intersection, or draw the graph of $y = x^3 \times 2^{-x} - 1$ and find where it intersects the x axis.

Maximum = (4.33, 4.04), Range is $0 \leq f(x) \leq 4.04$, $x = 1.37$ or 9.94

Graphs of Functions

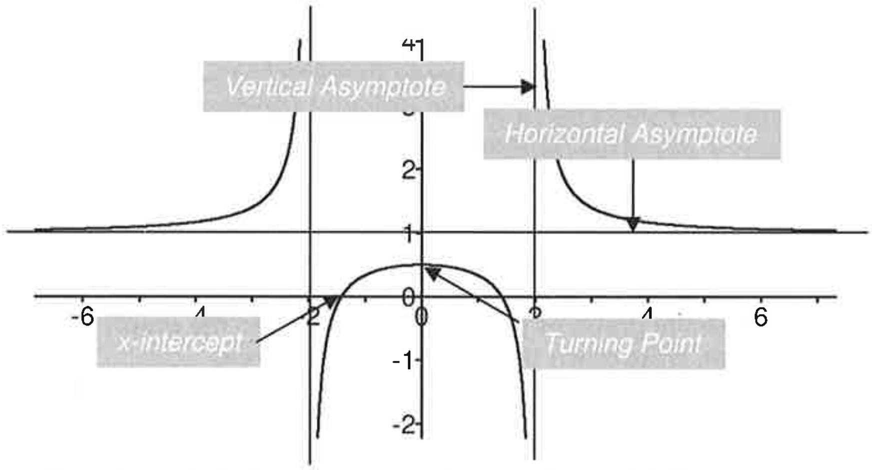
A graph is an excellent tool for interpreting a function. From a graph we can see when the function is increasing or decreasing, what the range of the function is, where it cuts the axes and so on. Therefore it is important to be able to sketch and understand graphs of different types of functions. Remember that your calculator can be of great benefit, and you should fully understand its graphing functions; but you must also be able to sketch graphs without a calculator – see the section below.

Domain and range on a graph:



This graph shows that the function has a range $f(x) \geq -1$

Graphing terms:



- The vertical asymptote is caused by x-values for which the function would be undefined (ie domain restrictions)
- The horizontal asymptote indicates that the value of the function for very large x-values (both positive and negative) tends to a limit
- Turning points can be maximums or minimums. At these points the gradient of the graph is zero.
- x- and y-intercepts can be calculated by setting (respectively) the y and x values in the function to 0.

Transformations of graphs: You should be able to sketch the graphs of the basic functions $y = x^2$, $y = x^3$, $y = 1/x$, $y = a^x$, $y = \log x$. The effect of simple numerical changes to these functions (involving additions, multiplications and minus signs) results in specific, simple transformations, thus extending the number of functions which can be easily sketched.

The graph transformations you need to know are:

Change to function	Transformation
$y = f(x) + a$	Move graph upwards by *a* units
$y = f(x + a)$	Move graph to the *left* by *a* units
$y = af(x)$	Stretch graph vertically by scale factor *a*
$y = f(ax)$	Stretch graph horizontally by scale factor *1/a*
$y = -f(x)$	Reflect graph in *x*-axis
$y = f(-x)$	Reflect graph in *y*-axis

Transformations in the *x* direction always do the opposite of what you expect!

For example, $y = (x-1)^2 + 2$ will move the graph of $y = x^2$ to the right by 1 and up by 2, that is, a translation of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$y = -\dfrac{3}{x+2}$ is a multiple transformation of $y = \dfrac{1}{x}$. To obtain the correct order of transformations, consider what order you would work out the expression if you put in a value for *x*. This would be:

- Add 2 to *x*
- Multiply the function by 3
- Change sign

Be aware of the difference between, say, adding 2 to the *x* part of the function, and adding 2 to the *whole* function.

The equivalent transformations are:

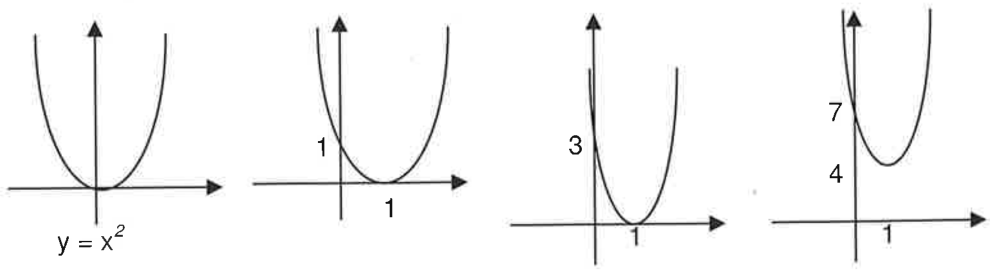
- Move left 2 units
- Stretch by 3 in the *y* direction
- Reflect in the *x*-axis

The graph of $f^{-1}(x)$: Consider the graph of $y = x^2$ (which represents the function $f(x) = x^2$). When $x = 3$, $y = 9$ (ie $f(3) = 9$). The graph of the inverse function is $y = \sqrt{x}$, and when $x = 9$, $y = 3$. *Any* point (a, b) on the graph of $f(x)$ becomes (b, a) on the graph of $f^{-1}(x)$. This represents a reflection in the line $y = x$.

- The graph of $f^{-1}(x)$ is always the graph of $f(x)$ reflected in the line $y = x$.

YOU SOLVE

The diagrams show how the graph of $y = x^2$ is transformed to the graph of $y = f(x)$ in three steps. For each diagram, write down the equation of the curve.



$y = (x-1)^2$, $y = 3(x-1)^2$, $y = 3(x-1)^2 + 4$

Linear Functions

In a linear function, the function increases (or decreases) at a constant rate. Its graph is a straight line.

Linear Functions are in the "presumed knowledge", rather that in the Standard syllabus itself.

Example: Cost of printing programmes against number of programmes printed.
Equation: $f(x) = ax + b$ where a and b are constants.
Gradient: The *gradient* of the line is its "steepness." A gradient of 3 means that y is increasing 3 times faster than x. The gradient is calculated by choosing two points and dividing the change in y by the change in x.

Horizontal lines have gradient 0. Vertical lines have an infinite gradient. Lines angled from bottom left to top right have positive gradients, others have negative gradients.

Formulae

The gradient between two points (x_1, y_1) and (x_2, y_2) can be calculated as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

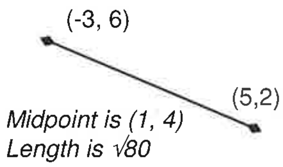
Parallel lines: $m_1 = m_2$

Perpendicular lines: $m_1 m_2 = -1$

Midpoint of two points is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Distance between two points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint, distance between two points:



The midpoint of two points can be found by calculating the x-coordinate halfway between the x-coordinates of the two points, and the same for the y-coordinate. The distance between two points is calculated using Pythagoras' Theorem. Although the formulae are shown on the right, this means there are a lot of formulae to remember. It is often better to draw a sketch and work from that.

Drawing a line from its equation:

- If the equation is of the form $y = ax + b$, substitute 2 or 3 values for x and work out the corresponding y values.
- If the equation is of the form $ax + by = c$, it is easier to put x equal to 0 and work out y, then put y equal to 0 and work out x. This gives the two points where the line crosses the axes.
- To *sketch* the graph of $y = ax + b$, remember that b is the y-intercept and a is the gradient.

A useful line to remember is that with equation $x + y = a$. This line always passes through $(0, a)$ and $(a, 0)$

Working out the equation from the graph:

	There are two formulae you can use.
1. Calculate the gradient.	
2a. If using the first formula, replace m with the gradient, then substitute a point for x and y. 3a. Calculate c and then put this back into the equation.	$y = mx + c$
2b. If using the second formula, replace m with the gradient then substitute the point for x_1 and y_1. 3b. Rearrange and simplify to get the equation.	$(y - y_1) = m(x - x_1)$

The points P, Q have coordinates P(3, 0), Q(-3, 7). Find the equation of the line which is perpendicular to PQ and passes through the point P. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.
(You have a point on the line, you just need the gradient. Use the formula for gradients of perpendicular lines).

$-6x + 7y + 18 = 0$

YOU SOLVE

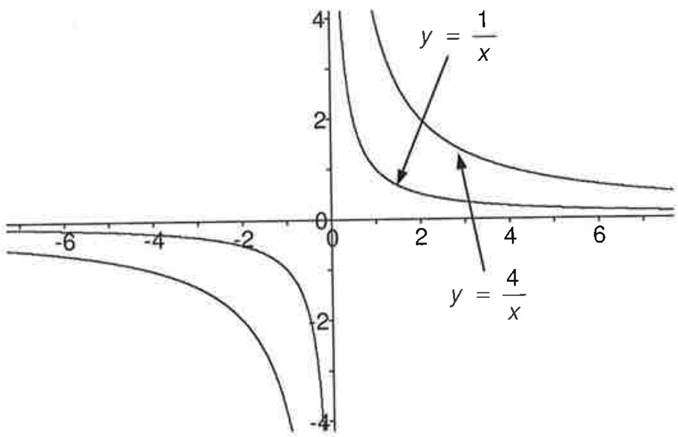
Reciprocal Functions

In the reciprocal function $f(x) = \frac{a}{x}$, where a is a constant, the function *decreases* as the x values increase. Specifically, if an x value is multiplied by any number, the y value will be divided by the same number.

Example: The time taken to fly a fixed distance against the speed. (If the speed doubles, the time halves).

Graph: The diagram shows the graphs of two reciprocal functions. They have similar shapes. Each graph is in two sections, with the y-axis being a vertical asymptote. Since they are also self-reflections about $y = x$ this means that a reciprocal function is its own inverse. For example $f(x) = \frac{12}{x} \Rightarrow f^{-1}(x) = \frac{12}{x}$.

This is easily seen if 2 is put into the function: $\frac{12}{2} = 6$ then $\frac{12}{6} = 2$.



YOU SOLVE

a) Given that $f(x) = 1 - x$ and $g(x) = \frac{1}{x}$, find $h(x) = (f \circ g \circ f)(x)$, simplifying your answer.

b) Prove that $h(x)$ is a self-inverse function.

For (b), either find the inverse function or prove $(h \circ h)(x) = x$.

$\frac{x}{x-1}$

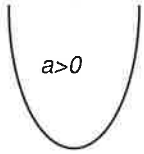
YOU SOLVE

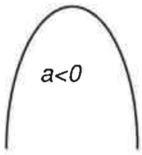
The graph of $y = \frac{1}{x}$ is given the following transformations: reflection in the x-axis followed by a translation $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Write down the equation of the new graph and its horizontal and vertical asymptotes

$-\frac{1}{x} - 2, \ y = -2, \ x = 0$

Quadratic Functions

Quadratic functions occur in many different situations. You should be completely familiar with the connections between the functions and their graphs, and with the methods for solving quadratic equations.





Equation: $f(x) = ax^2 + bx + c$

Graph: All quadratic graphs are parabolas, the sign of a determining "which way up." In the form shown above, we can say which way up the graph is and where the y-intercept is. For example, the graph of $y = x^2 + 3x - 4$ cuts the y-axis at (0, -4) and is in the shape of a U. The graph is always symmetrical about the vertical line passing through the vertex (turning point), a fact which can often be used when answering questions about the graph.

Factorisation: Quadratics come in different forms:

- $ax^2 + bx = x(ax + b)$ eg: $2x^2 - 6x = 2x(x - 3)$
- $x^2 - a^2 = (x - a)(x + a)$ eg: $x^2 - 49 = (x - 7)(x + 7)$
- When all three terms of a quadratic are present, if it factorises, it will factorise into two brackets. Look for two numbers which multiply to give c and add to give b.
  $x^2 - 3x - 4 = (x - 4)(x + 1)$ *(Because $-4 \times 1 = -4$, $-4 + 1 = -3$)*
- If $a \neq 1$, first take out the number multiplying x as a factor.
  $2x^2 - 14x + 24 = 2(x^2 - 7x + 12) = 2(x - 4)(x - 3)$

In its factorised form, the equation reveals more information about the graph. If the equation factorises to $(x - p)(x - q)$ then the points (p, 0) and (q, 0) are the x-intercepts, ie the values of x^2 where the function equals zero.

Completing the square: This method gives us a third form of the quadratic function. Method and example are shown below.

1. For $x^2 + bx + c$ start by writing $(x + d)^2$ where $d = b \div 2$.	$x^2 + 6x + 7$ $(x + 3)^2$ ....
2. Now write down $-d^2$.	$(x + 3)^2 - 9$ ....
3. Write down c and simplify.	$(x + 3)^2 - 9 + 7$ $= \underline{\mathbf{(x + 3)^2 - 2}}$
For quadratics where $a \neq 1$, start by taking a out as a common factor. Forget about it whilst completing the square. Multiply it back at the end.	$2x^2 - 6x - 4$ $= 2(x^2 - 3x - 2)$ $= 2((x - 1.5)^2 - 2.25 - 2)$ $= 2((x - 1.5)^2 - 4.25)$ $= \underline{\mathbf{2(x - 1.5)^2 - 8.5}}$

In this form, the function can be seen to be a transformation of $y = x^2$. In the first example above, the transformation is a translation of $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$. Since the vertex of $y = x^2$ is (0, 0), the vertex of the new quadratic will be (-3, -2). In general, the completed square form is always:

$f(x) = a(x - h)^2 + k$ and this gives a vertex of (h, k).

Solving Quadratic Equations

$x^2 = 25$
$x = \pm 5$

Except for the simplest form of quadratic equation shown on the left the first move is **always collect together terms on the left with 0 on the right.**

Example: Solve the equation
$2x^2 - 4x = x^2 - 3$

$x^2 - 4x + 3 = 0$
$(x-3)(x-1) = 0$
$x = 3$ or 1

Factorisation: If the quadratic expression factorises, this is the simplest method of solution. Make sure you understand the connection between the factors and the *x*-intercepts (see previous section) since questions can link the equation to the graph.

Example: Solve the equation
$2x^2 - 4x = x + 2$

$2x^2 - 5x - 2 = 0$

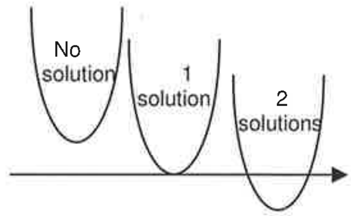
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times (-2)}}{2 \times 2}$$

$$x = \frac{5 \pm \sqrt{41}}{4}$$

$\therefore x = 2.851$ or -0.351

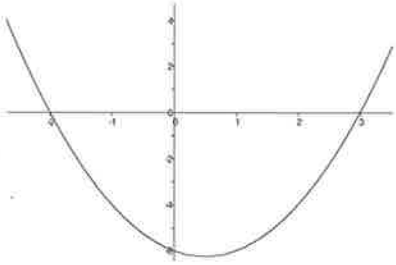
Formula: *All* quadratics can be solved using the formula, although it is most useful when the expression does not factorise.The solution of $ax^2 + bx + c = 0$ is: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It is the $\pm$ which leads to the two possible solutions. 🖩 You will find a program to solve quadratic equations a helpful tool.

Be careful to substitute correctly, particularly when there are minus signs around. Follow the example on the left carefully.



The solutions to a quadratic equation are the points where the graph crosses the *x*-axis. This can lead to 0, 1 or 2 solutions. These correspond to values of $b^2 - 4ac$ which are <0, =0 and >0 respectively. $b^2 - 4ac$ is called the *discriminant* since it discriminates between the number of solutions.

The diagram shows part of the graph with equation $y = x^2 + px + q$. The graph cuts the *x*-axis at –2 and 3. Find the values of *p* and *q*.



If the graph cuts the *x*-axis at –2 and 3, this tells us the two factors which form the equation must be $(x + 2)$ and $(x - 3)$

So the equation is $(x + 2)(x - 3) = 0$ and this multiplies out to give $x^2 - x - 6 = 0$.

Thus, $p = -1, q = -6$

The quadratic equation $3x^2 + 2px + 3 = 0$, $p > 0$, has exactly one solution for *x*. Find the value of *p*.

When questions refer to the number of solutions, think about the discriminant. In this case, the discriminant must equal 0.

$a = 3$, $b = 2p$, $c = 3$, so $(2p)^2 - 4 \times 3 \times 3 = 0$
$4p^2 - 36 = 0$
$p^2 = 9$ So, $p = 3$ (since $p > 0$)

YOU SOLVE

Express $f(x) = x^2 - 4x + 9$ in the form $f(x) = (x - h)^2 + k$. Hence, or otherwise, write down the coordinates of the vertex of the parabola with equation $y = x^2 - 4x + 9$.

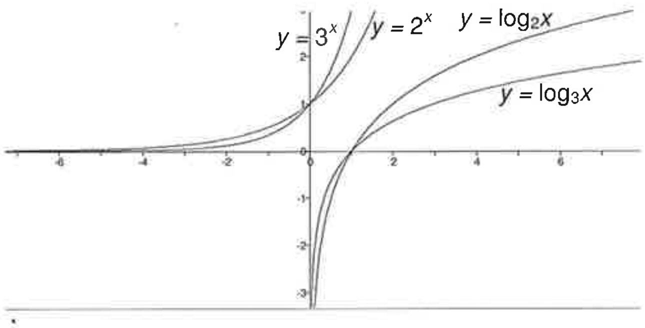
$f(x) = (x - 2)^2 + 5$, $(2, 5)$

Exponential and Logarithmic Functions

Equations: $f(x) = a^x$, $x \in$ Rational Numbers
$f(x) = \log_a x$, $x > 0$

Notice the domains. For a^x, x is restricted to rational numbers to make the working easier. However, it is not possible to find the logarithm of a negative number.

Graphs: In the same way that all quadratics have the same shape, all exponential and logarithmic curves are pretty much the same. The diagram on the right shows $y = 2^x$, $y = 3^x$, $y = \log_2 x$ and $y = \log_3 x$. The log functions are the inverses of the exponential functions, so their graphs are reflections of each other in the line $y = x$.



Questions draw on your knowledge of the laws of indices and logarithms. You must be familiar with the following rules:

- If $a^x = b$, then $x = \log_a b$ (eg: $2^3 = 8$, so $\log_2 8 = 3$)
- $x = \log_a a^x$ (This is similar to $x = \sqrt{x^2}$)
- $x = a^{\log_a x}$ (This is similar to $x = (\sqrt{x})^2$)

e^x and ln$x$: The number e is, like π, given a letter because it is irrational and hence impossible to write accurately using decimals. It is approximately 2.718. The functions e^x and e^{-x} are important because they are used to model situations where the rate of growth or decay of the quantity x is dependent on the value of x at any time. Typical applications are population growth and radioactive decay. The inverse of e^x is $\ln x$, short for $\log_e x$.

A group of ten monkeys is introduced to a zoo. After t years the number of monkeys, N, is modelled by $N = 10e^{0.3t}$.

a) How many monkeys are there after 2 years?

b) How long will it take for the number of monkeys to reach 50?

For (a) all we need to do is substitute $t = 2$. So $N = 10e^{(0.3 \times 2)} = 18.22$. There are <u>18 monkeys</u>

In part (b) we are asked to substitute $N = 50$. So, $50 = 10e^{0.3t}$. To find t, we need to bring the power down to "ground level", but we must divide by 10 first:

$$5 = e^{0.3t} \Rightarrow \ln 5 = \ln e^{0.3t} \Rightarrow \ln 5 = 0.3t \Rightarrow t = 5.36.$$

We can either give the answer as <u>5.36 years</u> or say <u>about 6 years.</u>

Find the domain of the function $f(x) = \sqrt{\ln(x - 3)}$

<u>$x \geq 4$</u>

YOU SOLVE

A population of bacteria is growing at the rate of 2.1% per minute. How long, to the nearest minute, will it take the population to double?

One of the features of exponential functions is that the time to double (or treble, or halve...) is the same, whatever value you start at. So, suppose we start the population at 100. We need to solve: $200 = 100 \times 1.021^t$ Now divide by 100, then use logs to solve.

<u>34 minutes</u>

YOU SOLVE