Properties of Logarithms

Topics

- The change of base formula.
- Logarithms of products, quotients, and powers.
- Applications of logarithms.

Theorems and Properties

- The **change of base formula** is $\log_a x = \dfrac{\log_{10} x}{\log_{10} a} = \dfrac{\ln x}{\ln a}$.
- $\log_a (uv) = \log_a u + \log_a v$
- $\log_a \left(\dfrac{u}{v}\right) = \log_a u - \log_a v$
- $\log_a u^n = n \log_a u$

The Rule of 72

The doubling time for a deposit earning $r\,\%$ interest compounded *annually* is $\dfrac{72}{r}$ years.

Note: IB DP SL Mathematics Students are not required to know Rule 72.

Example 1: Calculating Logarithms

- $\log_4 (25) = \dfrac{\log_{10} 25}{\log_{10} 4} \approx \dfrac{1.39794}{0.60206} \approx 2.32$

- $\log_4 (25) = \dfrac{\ln 25}{\ln 4} \approx \dfrac{3.21888}{1.38629} \approx 2.32$

Notice that the 2 answers agree.

Example 2: Simplifying Logarithms

One of the most important properties of logarithms is that the logarithms of products are the sums of the logarithms. Similarly, the logarithms of quotients are the differences of the logarithms.

- $\ln 2 + \ln 3 = \ln\left[(2)(3)\right] = \ln 6$

- $\ln \dfrac{2}{27} = \ln 2 - \ln 27$

- $\log_{10} 49 = \log_{10}\left(7^2\right) = 2\log_{10} 7$

Example 3: Writing a Product as a Sum

Logarithms play a major role in calculus by converting products to sums.

- $\log_4\left(5x^3y\right) = \log_4 5 + \log_4 x^3 + \log_4 y = \log_4 5 + 3\log_4 x + \log_4 y$
- $\ln\frac{\sqrt{3x-5}}{7} = \ln\left[\frac{(3x-5)^{1/2}}{7}\right] = \ln\left(3x-5\right)^{1/2} - \ln 7 = \frac{1}{2}\ln\left(3x-5\right) - \ln 7$

Example 4: The Rule of 72

The rule of 72 is a convenient tool for estimating the doubling time for an investment earning interest compounded annually. Suppose a bank pays 8% interest, compounded annually. Use the rule of 72 to approximate the doubling time.

Solution

The rule of 72 says that the doubling time is approximately $\frac{72}{8} = 9$ years. If you use the formula for compound interest, you will see that this approximation is excellent.

Study Tips

- Calculators generally have 2 buttons for logarithms: base 10 (common logarithm) and base e (natural logarithm). Use the change of base formula for other bases.
- The rule of 72 is for compound interest, as contrasted with the rule of 70 for continuous compounding.

Pitfalls

- There is no formula for the logarithm of a sum or difference. In particular, $\log_a\left(u+v\right) \neq \log_a u + \log_a v$.
- Be careful of domain changes when using properties of logarithms. For example, the domain of $y = \ln x^2$ is all $x \neq 0$, whereas the domain of $y = 2\ln x$ is all $x > 0$.
- The notation for logarithms can be confusing. Note that $\left(\ln x\right)^n \neq n\ln x$.

Problems

1. Evaluate the following logarithms using the change of base formula. Round your answer to 3 decimal places.

a. $\log_3 7$

b. $\log_{15} 1460$

2. Rewrite the following expressions in terms of $\ln 4$ and $\ln 5$.

a. $\ln 20$

b. $\ln \frac{5}{64}$

3. Use the properties of logarithms to rewrite and simplify the following logarithmic expressions

a. $\log_4 8$.

b. $\ln\left(5e^6\right)$.

4. Use the properties of logarithms to expand the expression $\ln\sqrt{z}$ as a sum, difference, and/or constant multiple of logarithms.

5. Condense the expression $\ln x - 3\ln(x+1)$ to the logarithm of a single quantity.

6. Find the exact value of $\log_3 9$ without using a calculator.

7. Find the exact value of $\ln e^3 - \ln e^7$ without using a calculator.

Exponential and Logarithmic Equations

Topics

- Equations involving logarithms and exponents.
- Approximate solutions.
- Applications.

Example 1: Solving an Exponential Equation

Solve the equation $4e^{2x} - 3 = 2$.

Solution

We use the properties of logarithms and exponents to solve for x.

$$\begin{aligned}
4e^{2x}-3&=2\\
4e^{2x}&=5\\
e^{2x}&=\frac{5}{4}\\
\ln e^{2x}&=\ln\left(\frac{5}{4}\right)\\
2x&=\ln\left(\frac{5}{4}\right)\\
x&=\frac{1}{2}\ln\left(\frac{5}{4}\right)\approx 0.11
\end{aligned}$$

Example 2: Solving a Logarithmic Equation

You should be careful to check your answers after solving an equation. In this example, you will see that one of the "solutions" is not valid. The appearance of these "extraneous" solutions occurs frequently with logarithmic equations because the domain of the logarithmic function is restricted to positive real numbers.

Solve the equation $\ln(x-2)+\ln(2x-3)=2\ln x$.

Solution

Notice how we use the properties of logarithms to simplify both sides and remove the logarithms.

$$\begin{aligned}
\ln(x-2)+\ln(2x-3)&=2\ln x\\
\ln\left[(x-2)(2x-3)\right]&=\ln x^2\\
(x-2)(2x-3)&=x^2\\
2x^2-7x+6&=x^2\\
x^2-7x+6&=0\\
(x-6)(x-1)&=0
\end{aligned}$$

There are 2 solutions to this quadratic equation: $x=1$ and $x=6$. However, $x=1$ is an extraneous solution because it is not in the domain of the original expression. Therefore, the only solution is $x=6$.

Example 3: Approximating the Solution of an Equation

In this example, we are forced to use a computer or graphing calculator to approximate the solution of the equation. Approximate the solution to the equation $\ln x = x^2-2$.

Solution

Write the equation as a function: $f(x)=\ln x - x^2+2$. The zeros of this function are the solutions to the original equation. Using a graphing utility, you see that the graph has 2 zeros, approximately 0.138 and 1.564. Note that it is impossible to find the exact solutions to the equation.

Study Tips

- There many ways to write an answer. For instance, in the first example, you could have written the answer as $\frac{1}{2}\ln\left(\frac{5}{4}\right)$, $\ln\left(\frac{5}{4}\right)^{1/2}$, $\ln\sqrt{\frac{5}{4}}$, $\ln\frac{\sqrt{5}}{2}$, or $\frac{1}{2}\left[\ln 5 - \ln 4\right]$. All of these are correct.
- Recall that the logarithm of a sum is not the sum of the logarithms. That is, the equation $\log_{10}\left(x+10\right) \neq \log_{10} x + \log_{10} 10$ for all values of x. However, you can show that this equation does have a solution: $x = 10/9$.

Pitfalls

- It's important to distinguish between exact and approximate answers. For instance, the exact answer to the equation $5 + 2\ln x = 4$ is $x = e^{-1/2} = \frac{1}{\sqrt{e}}$. This is approximately equal to 0.61, correct to 2 decimal places.

- Be careful of extraneous solutions when solving equations involving logarithms and exponents. Make sure that you check your answers in the original equation.

Problems

1. Solve the following exponential equations.

a. $4^x = 16$

b. $\left(\frac{1}{8}\right)^x = 64$

2. Solve the following logarithmic equations.

a. $\ln x = -7$

b. $\ln(2x-1) = 5$

3. Solve the following exponential equations. Round your result to 3 decimal places.

a. $8^{3x} = 360$

b. $\left(1+\frac{0.10}{12}\right)^{12t} = 2$

c. $e^{2x} - 4e^x - 5 = 0$

d. $e^x = e^{x^2-2}$

4. Solve the following logarithmic equations. Round your result to 3 decimal places.

a. $\log_5(3x+2) = \log_5(6-x)$

b. $\log_4 x - \log_4(x-1) = \frac{1}{2}$

5. Use a graphing utility to approximate the solution of the equation $\log_{10} x = x^3 - 3$, accurate to 3 decimal places.

Answers:

Properties of Logarithms

1. **a.** $\log_3 7 = \frac{\ln 7}{\ln 3} \approx \frac{1.9459}{1.0986} \approx 1.771$

b. $\log_{15} 1460 = \frac{\ln 1460}{\ln 15} \approx \frac{7.2862}{2.7081} \approx 2.691$

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2. **a.** $\ln 20 = \ln(4 \cdot 5) = \ln 4 + \ln 5$

b. $\ln \frac{5}{64} = \ln 5 - \ln 64 = \ln 5 - \ln 4^3 = \ln 5 - 3\ln 4$

3. **a.** $\log_4 8 = \ln_4 2^3 = 3\ln_4 2 = 3\ln_4 (4)^{1/2} = 3\left(\frac{1}{2}\right)\ln_4 4 = \frac{3}{2}$

b. $\ln\left(5e^6\right) = \ln 5 + \ln e^6 = \ln 5 + 6\ln e = \ln 5 + 6$

4. $\ln\sqrt{z} = \ln z^{1/2} = \frac{1}{2}\ln z$

5. $\ln x - 3\ln(x+1) = \ln x - \ln\left(x+1\right)^3 = \ln \frac{x}{\left(x+1\right)^3}$

6. $\log_3 9 = \log_3 3^2 = 2\log_3 3 = 2(1) = 2$

7. $\ln e^3 - \ln e^7 = 3\ln e - 7\ln e = 3 - 7 = -4$

Exponential and Logarithmic Equations

1. **a.** $4^x = 16 = 4^2 \Rightarrow x = 2$

b. $\left(\frac{1}{8}\right)^x = 64 \Rightarrow 8^{-x} = 8^2 \Rightarrow -x = 2 \Rightarrow x = -2$

2. **a.** $\ln x = -7 \Rightarrow x = e^{-7}$

b. $\ln(2x-1) = 5 \Rightarrow 2x - 1 = e^5 \Rightarrow 2x = 1 + e^5 \Rightarrow x = \frac{1}{2}\left(1 + e^5\right)$

3. **a.** $8^{3x} = 360 \Rightarrow \ln 8^{3x} = 3x\ln 8 = \ln 360 \Rightarrow x = \frac{\ln 360}{3\ln 8} \approx 0.944$

b. $\left(1 + \frac{0.10}{12}\right)^{12t} = 2$

$$\left(\frac{12.1}{12}\right)^{12t} = 2$$

$$12t\ln\frac{12.1}{12} = \ln 2$$

$$t = \frac{1}{12}\frac{\ln 2}{\ln\left(12.1/12\right)} \approx 6.960$$

c. $e^{2x} - 4e^{x} - 5 = 0$

$e^{x} = 5 \Rightarrow x = \ln 5 \approx 1.609$

$e^{x} = -1$, impossible.

d. $e^{x} = e^{x^2-2} \Rightarrow x = x^2 - 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, \ -1$

4. **a.** $\log_5(3x+2) = \log_5(6-x)$

$3x + 2 = 6 - x$

$4x = 4$

$x = 1$

b. $\log_4 x - \log_4(x-1) = \dfrac{1}{2}$

$\log_4\left(\dfrac{x}{x-1}\right) = \dfrac{1}{2}$

$\dfrac{x}{x-1} = 4^{1/2} = 2$

$x = 2(x-1) = 2x - 2$

$x = 2$

5. Using a graphing utility, graph the function $y = \log_{10} x - x^3 + 3$. You will obtain the 2 zeros, $x \approx 1.469$ and $x \approx 0.001$.