Uses of Exponential and Logarithmic Functions

Topics in This Lesson

- The change of base formula.
- Solving equations that contain exponential and logarithmic functions.
- Solving problems that involve continuous compounded interest (the pert formula).

Definitions and Formulas

change of base formula: For any positive numbers M, b, and c, where $b \neq 1$ and $c \neq 1$,

$$\log_b M = \frac{\log_c M}{\log_c b}.$$

pert formula: $A = Pe^{rt}$, where A is the amount of money in your account at any time t, P is the amount of principal (the amount you put in initially, or when t = 0), r is the interest rate, and t is time (in years). This formula only applies when the interest on the account is compounded continuously.

Examples

Example 1

Solve $3^{2x+1} = 144$.

$$\begin{aligned} 3^{2x+1} &= 144 \\ \log_3(3^{2x+1}) &= \log_3(144) \\ (2x+1)\log_3(3) &= \log_3(144) \\ (2x+1)\cdot 1 &= \log_3(144) \\ 2x+1 &= \log_3(144) \\ 2x &= \log_3(144) - 1 \\ x &= \frac{\log_3(144)-1}{2} \end{aligned}$$

If we want to find an approximation of this value, we need to convert it to either a common logarithm (log base 10) or a natural logarithm (log base e), as most calculators only have buttons for these 2 types of logarithms. Let's choose to convert this to common logarithms (which are often written without a subscript). Thanks to the change of base formula, we have

$$x = \frac{\log_3(144)-1}{2}$$

$$= \frac{\frac{\log 144}{\log 3} - 1}{2}.$$

Now we can carefully use a calculator to estimate this solution: $x \approx 1.76186$.

Example 2

Solve $2\log x + \log 4 = 2$.

$$\begin{aligned} 2\log x + \log 4 &= 2 \\ \log(x^2) + \log 4 &= 2 \\ \log(4x^2) &= 2 \\ 10^{\log(4x^2)} &= 10^2 \\ 4x^2 &= 100 \\ x^2 &= 25 \\ x &= \pm 5 \end{aligned}$$

Now we must check our answers, just in case we have brought in any extraneous solutions.

$$\begin{aligned} 2\log 5 + \log 4 &= 2 \\ \log 5^2 + \log 4 &= 2 \\ \log(5^2 \cdot 4) &= 2 \\ \log 100 &= 2 \\ 2 &= 2 \end{aligned}$$

So $x = 5$ is a solution.

$$2\log(-5) + \log 4 = 2$$

But this equation makes no sense! Remember, the domain of a log function is the set of positive real numbers, so $\log(-5)$ is not defined. Therefore, $x = -5$ is not a solution. That means our only solution is $x = 5$.

Example 3

An initial investment of $1000 is now valued at $1750.67. The interest rate is 8%, compounded continuously. How long has the money been invested in this account?

Since the interest is being compounded continuously, we use the pert formula: $A = Pe^{rt}$. The initial investment amount of $1000 is the principal, so $P = 1000$. At this time, the amount in the account is $1750.67, and that is what is represented by A, so $A = 1750.67$. Lastly, the interest rate is 8%. We convert this to the decimal number 0.08, so $r = 0.08$. We now plug all this information into the formula.

$$\begin{aligned} A &= Pe^{rt} \\ 1750.67 &= 1000e^{0.08t} \\ 1.75067 &= e^{0.08t} \\ \ln 1.75067 &= 0.08t \\ \frac{\ln 1.75067}{0.08} &= t \end{aligned}$$

Using a calculator, we find that $t \approx 6.99998$. So the money has been in the account for about 7 years.

Example 4

An accountant realizes that one of his client's bank accounts now has $3795 in it. He knows that the bank has been continuously compounding interest on the account for 30 years at a rate of 7.5%. But he doesn't know how much money the client had when he started the account. He also knows that the client hasn't touched this account since starting it 30 years ago. So how much money did the client use to start that account?

Since the interest is being compounded continuously, we know that the pert formula is the one to use. A is the amount of money in the account now, so $A = 3795$. P is the principal, the initial amount in the account, which is what we're trying to solve for. We know that $r = 0.075$. And t is the number of years the account has been active, so $t = 30$. Plugging all this into the pert formula gives us the below.

$$\begin{aligned} 3795 &= Pe^{(0.075)(30)} \\ 3795 &= Pe^{2.25} \\ 3795 &= P(9.4877) \\ P &= \frac{3795}{9.4877} \\ P &\approx 399.99 \end{aligned}$$

That means that the client started the account with about $400.

Common Errors

- Errors when using a calculator to estimate these values. Care should be taken in making these calculations.
- Failing to memorize the change of base formula correctly.

Study Tips

- Continue to practice using the properties of logarithms.

Problems

Solve the following equations.

1. $11^{x-8} - 5 = 42$
2. $7 \cdot 6^{3x} = 42$
3. $10e^{2x-10} - 4 = 70$
4. $5^{4x+1} = 100$
5. $\log_3(x^2 + 8) - \log_3 4 = 3$
6. $\ln(x+7) + \ln(x+3) = \ln 77$
7. $-3\log_5(x+1) = -12$
8. $\log(x^2 + 9) = \log(7x - 3)$

9. An initial investment of $2000 was made 5 years ago. The interest rate is 6.5%, compounded continuously. What is the balance in the account now?

10. An initial investment of $5000 is now valued at $9110.59. The interest rate is 5%, compounded continuously. How long has the money been invested in this account?

Answers:

1.

$$11^{x-8} - 5 = 42$$
$$11^{x-8} = 47$$
$$\log_{11}(11^{x-8}) = \log_{11} 47$$
$$x - 8 = \log_{11} 47$$
$$x = 8 + \log_{11} 47$$

If we wish to use a calculator to estimate this value, we can use the change of base formula to convert the logarithm to common logarithms.

$$x = 8 + \log_{11} 47 = 8 + \frac{\log 47}{\log 11} \approx 9.6056$$

2.

$$7 \cdot 6^{3x} = 42$$
$$6^{3x} = 6$$
$$3x = 1$$
$$x = \frac{1}{3}$$

3.

$$10e^{2x-10} - 4 = 70$$
$$10e^{2x-10} = 74$$
$$e^{2x-10} = 7.4$$
$$\ln(e^{2x-10}) = \ln 7.4$$
$$2x - 10 = \ln 7.4$$
$$2x = 10 + \ln 7.4$$
$$x = \frac{10 + \ln 7.4}{2}$$
$$x \approx 6.00074$$

4.

$$5^{4x+1} = 100$$
$$\log_5(5^{4x+1}) = \log_5 100$$
$$4x + 1 = \log_5 100$$
$$4x = \log_5 100 - 1$$
$$x = \frac{\log_5 100 - 1}{4}$$
$$x = \frac{\frac{\log 100}{\log 5} - 1}{4}$$
$$x \approx 0.4653$$

5.

$$\log_3(x^2+8)-\log_3 4=3$$
$$\log_3\left(\frac{x^2+8}{4}\right)=3$$
$$\frac{x^2+8}{4}=3^3$$
$$x^2+8=108$$
$$x^2=100$$
$$x=\pm 10$$

6.

$$\ln(x+7)+\ln(x+3)=\ln 77$$
$$\ln((x+7)(x+3))=\ln 77$$
$$(x+7)(x+3)=77$$
$$x^2+10x+21=77$$
$$x^2+10x-56=0$$
$$(x+14)(x-4)=0$$

So $x = -14$ and $x = 4$ are the possible solutions. But x cannot equal -14; that would give us the natural logarithm of a negative number when we plug it in. So the only solution is $x = 4$.

7.

$$-3\log_5(x+1)=-12$$
$$\log_5(x+1)=4$$
$$x+1=5^4$$
$$x+1=625$$
$$x=624$$

8.

$$\log(x^2+9)=\log(7x-3)$$
$$x^2+9=7x-3$$
$$x^2-7x+12=0$$
$$(x-4)(x-3)=0$$

So $x = 3$ and $x = 4$ are solutions. We can confirm both of them.

9.

$$A=Pe^{rt}$$
$$A=2000e^{.065(5)}$$
$$A=\$2768.06$$

10.

$$\begin{aligned}
A &= Pe^{rt} \\
9110.59 &= 5000e^{0.05t} \\
\frac{9110.59}{5000} &= e^{0.05t} \\
\ln\left(\frac{9110.59}{5000}\right) &= 0.05t \\
t &= \frac{\ln\left(\frac{9110.59}{5000}\right)}{0.05} \\
t &\approx 11.99999
\end{aligned}$$

So the money has been in the account for 12 years.