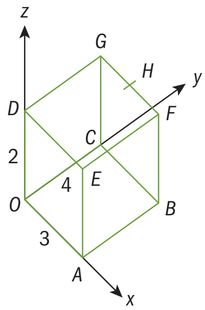
# 12 Vectors

## Skills check




**1** **a** $A = (3, 0, 0)$

**b** $B = (3, 4, 0)$

**c** $E = (3, 0, 2)$

**d** $F = (3, 4, 2)$

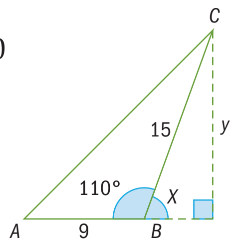
**e** $H = \left(\frac{3}{2}, 4, 2\right)$

**2** $x^2 = 3^2 + 6^2$

$= 9 + 36$

$= 45$

$x = \sqrt{45} \approx 6.71$




**3** **a** $X = 180 - 110 = 70$

$\cos X = \frac{z}{15} \Rightarrow z = 15\cos 70$

$\approx 5.13$

$\sin X = \frac{y}{15} \Rightarrow y = 15\sin 70$

$\approx 14.1$

$(AC)^2 = y^2 + (9+z)^2$

$= (14.1)^2 + (9 + 5.13)^2$

$AC = \sqrt{432.5}$

$= 20.8$

$= 21\,\text{cm}\,(\text{to the nearest centimetre})$

**b** Using the Cosine Rule

$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos(A\hat{B}C)$

$(9.7)^2 = (8.6)^2 + (3.1)^2 - 2(8.6)(3.1)\cos(A\hat{B}C)$

$A\hat{B}C = \cos^{-1}\left[\frac{(8.6)^2 + (3.1)^2 - (9.7)^2}{2(8.6)(3.1)}\right]$

$\approx 101.4°$

## Exercise 12A

**1** **a** $\mathbf{x} = -2\mathbf{i} + 3\mathbf{j}$

**b** $\mathbf{y} = 7\mathbf{j}$

**c** $\mathbf{z} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

**2** **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

**b** $\overrightarrow{CD} = \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix}$

**c** $\overrightarrow{EF} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

**3** **a** $= -3\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

**b** $= -2\mathbf{i} + 4\mathbf{j} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

**c** $= -3\mathbf{i} + 8\mathbf{j} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

**d** $= 6\mathbf{j} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

**e** $= -3\mathbf{i} - 6\mathbf{j} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

**4** **a** $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

**b** $\left|\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \approx 3.16$

**c** $|2\mathbf{i} + 5\mathbf{j}| = \sqrt{2^2 + 5^2} = \sqrt{29} \approx 5.39$

**d** $\left|\begin{pmatrix} 2.8 \\ 4.5 \end{pmatrix}\right| = \sqrt{(2.8)^2 + (4.5)^2} = 5.3$

**e** $|2\mathbf{i} - 5\mathbf{j}| = \sqrt{2^2 + (-5)^2} = \sqrt{29} \approx 5.39$

**5** **a** $\left|\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right| = \sqrt{3^2 + 2^2 + 5^2} = \sqrt{38} \approx 6.16$

**b** $\left|\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}\right| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \approx 5.10$

**c** $|2\mathbf{i} + 2\mathbf{j} + \mathbf{k}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

**d** $\left|\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix}\right| = \sqrt{(-3)^2 + 2^2 + 6^2} = \sqrt{49} = 7$

**e** $|\mathbf{j} - \mathbf{k}| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \approx 1.41$

## Exercise 12B

**1** **a** $= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, **b** $= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

**c** $= \begin{pmatrix} -6 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ -1 \end{pmatrix} = -3\mathbf{b}$

**d** $= \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \frac{1}{2}\mathbf{a}$

$\mathbf{e} = \begin{pmatrix}-9\\5\end{pmatrix} = s\mathbf{a} + t\mathbf{b}$ for some values of $s$ and $t$.

$= s\begin{pmatrix}2\\4\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix}$

We must have $s$ & $t$ so that

$-9 = 2s + 2t$ (1)

$5 = 4s - t$ (2)

$2 \times (2): 10 = 8s - 2t$ (3)

$(1) + (3): 1 = 10s$

$s = \frac{1}{10}$

from (2): $5 = \frac{4}{10} - t$

$t = \frac{4}{10} - \frac{50}{10}$

$= \frac{-46}{10}$

so $\mathbf{e} = \frac{1}{10}\mathbf{a} - \frac{46}{10}\mathbf{b}$

$\mathbf{f} = \begin{pmatrix}-5\\-8\end{pmatrix} = s\begin{pmatrix}2\\4\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix}$ for some $s$ & $t$.

We must have $s$ & $t$ so that

$-5 = 2s + 2t$ (1)

$-8 = 4s - t$ (2)

$2 \times (2): -16 = 8s - 2t$ (3)

$(1) + (3): -21 = 10s$

$s = \frac{-21}{10}$

from (2): $-8 = \frac{-84}{10} - t$

$t = 8 - \frac{84}{10}$

$= \frac{-2}{5}$

so $\mathbf{f} = \frac{-21}{10}\begin{pmatrix}2\\4\end{pmatrix} - \frac{2}{5}\begin{pmatrix}2\\-1\end{pmatrix}$

**2** **a** $\mathbf{a} = \begin{pmatrix}0.1\\0.7\end{pmatrix}$

$= \frac{1}{10}\begin{pmatrix}1\\7\end{pmatrix}$

$= \frac{1}{10}(\mathbf{i} + 7\mathbf{j})$

$\mathbf{a}$ is parallel to $\mathbf{i} + 7\mathbf{j}$ with $\frac{1}{10}$ the magnitude.

$\mathbf{b} = \begin{pmatrix}-1\\-7\end{pmatrix}$

$= -1(\mathbf{i} + 7\mathbf{j})$

$\mathbf{b}$ is parallel to $(\mathbf{i} + 7\mathbf{j})$ with opposite direction.

$\mathbf{c} = \begin{pmatrix}-0.05\\-0.03\end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{d} = \begin{pmatrix}-10\\70\end{pmatrix}$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{e} = 60\mathbf{i} + 420\mathbf{j}$

$= 60(\mathbf{i} + 7\mathbf{j})$

$\mathbf{e}$ is parallel to $(\mathbf{i} + 7\mathbf{j})$ with 60 times the magnitude

$\mathbf{f} = (6\mathbf{i} - 42\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

$\mathbf{g} = (-\mathbf{i} + 7\mathbf{j})$ is not parallel to $(\mathbf{i} + 7\mathbf{j})$

**3** **a** For parallel vectors, $\mathbf{r} = k\,\mathbf{s}$ for some $k$

$(4\mathbf{i} + t\mathbf{j}) = k(14\mathbf{i} - 12\mathbf{j})$

$4 = 14k$

$k = \frac{4}{14}$

$= \frac{2}{7}$

$t = -12k$

$= -12 \times \frac{2}{7}$

$= \frac{-24}{7}$

**b** For parallel vectors, $\mathbf{a} = k\,\mathbf{b}$ for some $k$

$\begin{pmatrix}t\\-8\end{pmatrix} = k\begin{pmatrix}7\\-10\end{pmatrix}$

$-8 = -10k$

$k = \frac{-8}{-10}$

$= \frac{4}{5}.$

so $t = 7k$

$= 7\left(\frac{4}{5}\right)$

$= \frac{28}{5}$

**4** For parallel vectors, $\mathbf{v} = k\,\mathbf{w}$ for some $k$

$t\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = k(5\mathbf{i} + \mathbf{j} + 5\mathbf{k})$

$-5 = k$

so $t = 5k$

$t = 5(-5)$

$= -25$

$8 = sk = (-5)s$

$s = \frac{-8}{5}$

**5** **a** $\overrightarrow{OG} = \mathbf{j} + \mathbf{k}$

**b** $\overrightarrow{BD} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$

**c** $\overrightarrow{AD} = -\mathbf{i} + \mathbf{k}$

**d** $\overrightarrow{OM} = \frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}$

**6** **a** $\overrightarrow{OG} = 4\mathbf{j} + 3\mathbf{k}$

**b** $\overrightarrow{BD} = -5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

**c** $\overrightarrow{AD} = -5\mathbf{i} + 3\mathbf{k}$

**d** $\overrightarrow{OM} = \frac{5}{2}\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

## Exercise 12C

**1** $\overrightarrow{OP}=\begin{pmatrix}7\\4\end{pmatrix}$ $\quad\overrightarrow{OQ}=\begin{pmatrix}2\\3\end{pmatrix}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}7\\4\end{pmatrix}=\begin{pmatrix}-5\\-1\end{pmatrix}$$

$$\overrightarrow{QP}=-\overrightarrow{PQ}=\begin{pmatrix}5\\1\end{pmatrix}$$

**2** $A=\begin{pmatrix}5\\1\end{pmatrix}$ $B=\begin{pmatrix}1\\-3\end{pmatrix}$ $C=\begin{pmatrix}-2\\4\end{pmatrix}$

**a** $\overrightarrow{AB}=B-A=\begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}-4\\-4\end{pmatrix}$

**b** $\overrightarrow{BA}=-\overrightarrow{AB}=\begin{pmatrix}4\\4\end{pmatrix}$

**c** $\overrightarrow{AC}=C-A=\begin{pmatrix}-2\\4\end{pmatrix}-\begin{pmatrix}5\\1\end{pmatrix}=\begin{pmatrix}-7\\3\end{pmatrix}$

**d** $\overrightarrow{CB}=B-C=\begin{pmatrix}1\\-3\end{pmatrix}-\begin{pmatrix}-2\\4\end{pmatrix}=\begin{pmatrix}3\\-7\end{pmatrix}$

**3** **a** $\overrightarrow{OP}=\begin{pmatrix}2\\-3\\5\end{pmatrix}=2\mathbf{i}-3\mathbf{j}+5\mathbf{k}$

**b** vector is $\begin{pmatrix}1\\-5\\6\end{pmatrix}=\mathbf{i}-5\mathbf{j}+6\mathbf{k}$

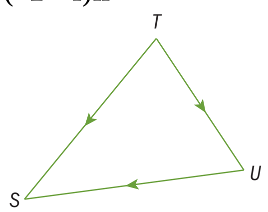
**c** vector is $\begin{pmatrix}1\\2\\-1\end{pmatrix}-\begin{pmatrix}2\\-3\\5\end{pmatrix}=\begin{pmatrix}-1\\5\\-6\end{pmatrix}=-\mathbf{i}+5\mathbf{j}-6\mathbf{k}$

**d** vector is $\begin{pmatrix}2\\-3\\5\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1\\-5\\6\end{pmatrix}=\mathbf{i}-5\mathbf{j}+6\mathbf{k}$

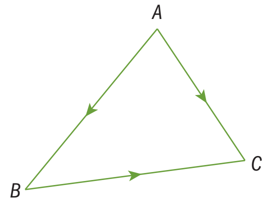
**4** $\overrightarrow{LM}=\overrightarrow{LN}+\overrightarrow{NM}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\begin{pmatrix}4\\-2\\-3\end{pmatrix}=\begin{pmatrix}5\\-4\\-3\end{pmatrix}$

**5** From the diagram, we see

$$\overrightarrow{US}=-\overrightarrow{TU}+\overrightarrow{TS}$$
$$=-(\mathbf{i}-4\mathbf{j}+2\mathbf{k})+(3\mathbf{i}+4\mathbf{j}-\mathbf{k})$$
$$=(-1+3)\mathbf{i}+(4+4)\mathbf{j}+(-2-1)\mathbf{k}$$
$$=2\mathbf{i}+8\mathbf{j}-3\mathbf{k}$$




**6** From the diagram,




$$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=0$$

$$\begin{pmatrix}1\\y\\-2\end{pmatrix}+\begin{pmatrix}2x\\-3\\z\end{pmatrix}-\begin{pmatrix}1\\4\\x+y\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$1+2x+1=0\Rightarrow 2+2x=0$ (1)

$y-3+4=0\Rightarrow y+1=0$ (2)

$-2+z+x+y=0\Rightarrow x+y+z-2=0$ (3)

(1) $\Rightarrow x=-1$

(2) $\Rightarrow y=-1$

(3) $\Rightarrow -2+z-2=0$

$z=4$

## Exercise 12D

**1** $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(-2\mathbf{i}+3\mathbf{j}-\mathbf{k})-(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})$
$$=(-2-1)\mathbf{i}+(3-(-2))\mathbf{j}+(-1-3)\mathbf{k}$$
$$=-3\mathbf{i}+5\mathbf{j}-4\mathbf{k}$$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=(4\mathbf{i}-7\mathbf{j}+7\mathbf{k})-(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$
$$=(4-1)\mathbf{i}+(-7-(-2))\mathbf{j}+(7-3)\mathbf{k}$$
$$=3\mathbf{i}-5\mathbf{j}+4\mathbf{k}$$

we see $\overrightarrow{AB}=-\overrightarrow{AC}$, so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. Since they contain a common point A, they must lie on the same line.

**2** **a** $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}5\\1\\5\end{pmatrix}-\begin{pmatrix}2\\3\\-3\end{pmatrix}=\begin{pmatrix}3\\-2\\8\end{pmatrix}$

**b** $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}8\\-1\\13\end{pmatrix}-\begin{pmatrix}2\\3\\-3\end{pmatrix}=\begin{pmatrix}6\\-4\\16\end{pmatrix}$

we see $\overrightarrow{AC}=2\overrightarrow{AB}$, so $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel. Since they contain a common point A, then A, B, & C are collinear.

**3** $\overrightarrow{P_1P_2}=\overrightarrow{OP}_2-\overrightarrow{OP}_1=\begin{pmatrix}-2\\1\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-3\\-1\\0\end{pmatrix}$

$\overrightarrow{P_1P_3}=\overrightarrow{OP}_3-\overrightarrow{OP}_1=\begin{pmatrix}-5\\0\\4\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}-6\\-2\\0\end{pmatrix}$

we see $\overrightarrow{P_1P_3}=2\overrightarrow{P_1P_2}$. Since they contain a common point, they are collinear.

Since $P_4$ collinear with $P_1$, $P_2$, $P_3$, we have

$\overrightarrow{P_1P_4}=k\overrightarrow{P_1P_2}$ for some $k\in R$

$\overrightarrow{P_1P_4}=\begin{pmatrix}2\\s\\t\end{pmatrix}-\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}1\\s-2\\t-4\end{pmatrix}$ for some $s$ & $t$

Now $\begin{pmatrix} 1 \\ s-2 \\ t-2 \end{pmatrix} = k\begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$

$1 = -3k \Rightarrow k = \frac{-1}{3}$

$s - 2 = -k \Rightarrow s = 2 - k = 2 + \frac{1}{3} = \frac{7}{3}$

$t - 2 = 0 \Rightarrow t = 2$

$\therefore P_4 = \left(2, \frac{7}{3}, 2\right)$

**4** $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}, \overrightarrow{OB} = x\mathbf{i}, \overrightarrow{OC} = \mathbf{i} - 2\mathbf{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x-3)\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (1-3)\mathbf{i} + (-2-4)\mathbf{j}$

$= -2\mathbf{i} - 6\mathbf{j}$

If A, B, C are collinear, $\overrightarrow{AB} = k\overrightarrow{AC}$ for some $k \in R$

$\therefore (x-3)\mathbf{i} - 4\mathbf{j} = k(-2\mathbf{i} - 6\mathbf{j})$

$\mathbf{j}$ components $\Rightarrow -4 = -6k \Rightarrow k = \frac{2}{3}$

so $x - 3 = 2k = \frac{-4}{3}$

$x = \frac{9}{3} - \frac{4}{3} = \frac{5}{3}$

so $\overrightarrow{AB} = \frac{-4}{3}\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (\mathbf{i} - 2\mathbf{j}) - \frac{5}{3}\mathbf{i} = \frac{-2}{3}\mathbf{i} - 2\mathbf{j}$

$\overrightarrow{AB} : \overrightarrow{BC} = \begin{pmatrix} \frac{-4}{3} \\ -4 \end{pmatrix} : \begin{pmatrix} \frac{-2}{3} \\ -2 \end{pmatrix}$

$= 2:1$

## Exercise 12E

**1** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$

Distance AB $= \sqrt{5^2 + (-2)^2}$

$= \sqrt{29}$

$\approx 5.39$

**2** $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 2 \end{pmatrix}$

Distance AB $= \sqrt{11^2 + 2^2 + 2^2} = \sqrt{129}$

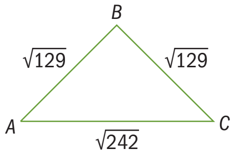
$\overrightarrow{AC} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ -3 \end{pmatrix}$

Distance AC $= \sqrt{13^2 + 8^2 + (-3)^2} = \sqrt{242}$

$\overrightarrow{BC} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -5 \end{pmatrix}$

Distance BC $= \sqrt{2^2 + 10^2 + 5^2} = \sqrt{129}$

Distance AB = Distance BC, so ABC is isosceles.




$\cos\left(C\hat{A}B\right) = \frac{129 + 242 - 129}{2\sqrt{129}\sqrt{242}}$

$C\hat{A}B = 46.8°$

**3** $|\mathbf{a}| = 7$, so $\sqrt{2^2 + (-3)^2 + t^2} = 7$

$4 + 9 + t^2 = 49$

$t^2 = 36$

$t = \pm 6$

**4** $\mathbf{a} = x\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$|\mathbf{a}| = \sqrt{x^2 + 3^2 + (-2)^2} = 22$

$x^2 + 9 + 4 = 484$

$x^2 = 471$

$x^2 = \pm\sqrt{471} = \pm 21.7$ (3s.f.)

**5** $|\mathbf{u}| = |\mathbf{v}|$, so

$a^2 + (-a)^2 + (2a)^2 = 2^2 + (-4)^2 + (-2)^2$

$a^2 + a^2 + 4a^2 = 4 + 16 + 4$

$6a^2 = 24$

$a^2 = 4$

$a = \pm 2$

**6** **a** $\mathbf{b} = 2\mathbf{a}$

Then $|\mathbf{a} + \mathbf{b}| = |3\mathbf{a}| = 3|\mathbf{a}| = 15$

**b** $\mathbf{b} = -3\mathbf{a}$

Then $|\mathbf{a} + \mathbf{b}| = |-2\mathbf{a}| = 2|\mathbf{a}| = 10$

**c** $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}| - |\mathbf{a} \cdot \mathbf{b}|$

$\mathbf{a}$ & $\mathbf{b}$ are perpendicular, so $\mathbf{a} \cdot \mathbf{b} = 0$

Hence $|\mathbf{a} + \mathbf{b}| = 5 + 12 = 17$

## Exercise 12F

**1** $\left|\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{1} = 1$

**2** $\left|\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right| = \sqrt{\frac{1}{3^2} + \frac{2^2}{3^2} + \frac{2^2}{3^2}}$

$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$

$= \sqrt{1} = 1$

**3** **a** $|4\mathbf{i} - 3\mathbf{j}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

So unit vector is $\frac{1}{5}(4\mathbf{i} - 3\mathbf{j}) = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

**b** $\left|\begin{pmatrix} 2 \\ -5 \end{pmatrix}\right| = \sqrt{2^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$

So unit vector is $\frac{1}{\sqrt{29}}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

**4** **a** $\left|4\mathbf{i}+\mathbf{j}+\sqrt{8}\mathbf{k}\right|=\sqrt{4^2+1+8}$

$=\sqrt{25}=5$

So unit vector is $\frac{1}{5}\left(4\mathbf{i}+\mathbf{j}+\sqrt{8}\mathbf{k}\right)$

**b** $\left|\begin{pmatrix}-1\\-5\\4\end{pmatrix}\right|=\sqrt{(-1)^2+(-5)^2+4^2}=\sqrt{42}$

So unit vector is $\frac{1}{\sqrt{42}}\begin{pmatrix}-1\\-5\\4\end{pmatrix}$

**5** $\overline{P_1P_2}=\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}2\\2\\-1\end{pmatrix}$

$\left|\overline{P_1P_2}\right|=\sqrt{2^2+2^2+(-1)^2}=\sqrt{9}=3$

So unit vector is $\frac{1}{3}\begin{pmatrix}2\\2\\-1\end{pmatrix}$

**6** $\left|a\mathbf{i}+2a\mathbf{j}\right|=\sqrt{a^2+(2a)^2}=\sqrt{5a^2}$

$=\sqrt{5}a$

Now $\sqrt{5}a=1$, so $a=\frac{1}{\sqrt{5}}$

**7** $\left|2\mathbf{i}-\mathbf{j}\right|=\sqrt{2^2+(-1)^2}$

$=\sqrt{5}$

So unit vector is $\frac{1}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})$

Vector of magnitude 5 is $\frac{5}{\sqrt{5}}(2\mathbf{i}-\mathbf{j})=\sqrt{5}(2\mathbf{i}-\mathbf{j})$

**8** $\left|\begin{pmatrix}-1\\-3\\2\end{pmatrix}\right|=\sqrt{(-1)^2(-3)^2+2^2}=\sqrt{14}$

unit vector is $\frac{1}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

and vector magnitude 7 is $\frac{7}{\sqrt{14}}\begin{pmatrix}-1\\-3\\2\end{pmatrix}=\frac{\sqrt{14}}{2}\begin{pmatrix}-1\\-3\\2\end{pmatrix}$

**9** $\left|\begin{pmatrix}-2\\0\\2\end{pmatrix}\right|=\sqrt{(-2)^2+2^2}=\sqrt{8}=2\sqrt{2}$

So unit vector $=\frac{1}{2\sqrt{2}}\begin{pmatrix}-2\\0\\2\end{pmatrix}$

Vector of magnitude $\sqrt{2}$ is $\frac{1}{2}\begin{pmatrix}-2\\0\\2\end{pmatrix}$

**10** $\left|\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}\right|=\sqrt{2^2\cos^2\theta+2^2\sin^2\theta}$

$=\sqrt{4\left(\cos^2\theta+\sin^2\theta\right)}$

$=2\sqrt{(1)}=2$

So unit vector is $\frac{1}{2}\begin{pmatrix}2\cos\theta\\2\sin\theta\end{pmatrix}$ or $\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$

## Exercise 12G

**1** **a** $\mathbf{a}+\mathbf{b}=(2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})$

$=(2+3)\mathbf{i}+(-1+2)\mathbf{j}$

$=5\mathbf{i}+\mathbf{j}$

**b** $\mathbf{b}+\mathbf{c}=(3\mathbf{i}+2\mathbf{j})+(-\mathbf{i}+\mathbf{j})$

$=(3-1)\mathbf{i}+(2+1)\mathbf{j}$

$=2\mathbf{i}+3\mathbf{j}$

**c** $\mathbf{c}+\mathbf{d}=(-\mathbf{i}+\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$=(-1+3)\mathbf{i}+(1+3)\mathbf{j}$

$=2\mathbf{i}+4\mathbf{j}$

**d** $\mathbf{a}+\mathbf{b}+\mathbf{d}=(2\mathbf{i}-\mathbf{j})+(3\mathbf{i}+2\mathbf{j})+(3\mathbf{i}+3\mathbf{j})$

$=(2+3+3)\mathbf{i}+(-1+2+3)\mathbf{j}$

$=8\mathbf{i}+4\mathbf{j}$

**e** $\mathbf{a}-\mathbf{b}=(2\mathbf{i}-\mathbf{j})-(3\mathbf{i}+2\mathbf{j})$

$=(2-3)\mathbf{i}+(-1-2)\mathbf{j}$

$=-\mathbf{i}-3\mathbf{j}$

**f** $\mathbf{d}-\mathbf{b}+\mathbf{a}=(3\mathbf{i}+3\mathbf{j})-(3\mathbf{i}+2\mathbf{j})+(2\mathbf{i}-\mathbf{j})$

$=(3-3+2)\mathbf{i}+(3-2-1)\mathbf{j}$

$=2\mathbf{i}+0\mathbf{j}=2\mathbf{i}$

**2** **a** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-4\\5\end{pmatrix}=\begin{pmatrix}2&-4\\-3&+5\end{pmatrix}=\begin{pmatrix}-2\\2\end{pmatrix}$

**b** $\mathbf{b}-\mathbf{c}=\begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}=\begin{pmatrix}-4&-(-5)\\5&-(-3)\end{pmatrix}=\begin{pmatrix}1\\8\end{pmatrix}$

**c** $\frac{1}{2}(\mathbf{a}+\mathbf{c})=\frac{1}{2}\left[\begin{pmatrix}2\\-3\end{pmatrix}+\begin{pmatrix}-5\\-3\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}2&-5\\-3&+(-3)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-3\\-6\end{pmatrix}$

**d** $\mathbf{a}+3\mathbf{b}-\mathbf{c}=\begin{pmatrix}2\\-3\end{pmatrix}+3\begin{pmatrix}-4\\5\end{pmatrix}-\begin{pmatrix}-5\\-3\end{pmatrix}$

$=\begin{pmatrix}2+&3(-4)&-(-5)\\-3&3(5)&-(-3)\end{pmatrix}$

$=\begin{pmatrix}0\\15\end{pmatrix}$

**e** $3\mathbf{c} - 2\mathbf{b} + 5\mathbf{a} = 3\begin{pmatrix}-5\\-3\end{pmatrix} - 2\begin{pmatrix}-4\\5\end{pmatrix} + 5\begin{pmatrix}2\\-3\end{pmatrix}$

$= \begin{pmatrix}3(-5) - 2(-4) + 5(2)\\3(-3) - 2(5) + 5(-3)\end{pmatrix}$

$= \begin{pmatrix}-15 + 8 + 10\\-9 - 10 - 5\end{pmatrix}$

$= \begin{pmatrix}3\\-34\end{pmatrix}$

**3** **a** $\mathbf{a} + \mathbf{b} = (3\mathbf{i} - \mathbf{j} - 2\mathbf{k}) + (5\mathbf{i} - \mathbf{k})$

$= (3 + 5)\mathbf{i} + (-1)\mathbf{j} + (-2 - 1)\mathbf{k}$

$= 8\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

**b** $\mathbf{b} - 2\mathbf{a} = (5\mathbf{i} - \mathbf{k}) - 2(3\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= (5 - 6)\mathbf{i} - 2(-1)\mathbf{j} + (-1 - 2(-2))\mathbf{k}$

$= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

So $\mathbf{y} = \frac{1}{3}(19\mathbf{i} + 8\mathbf{j})$

**c** $2\mathbf{p} + \mathbf{z} = 0$

$2\begin{pmatrix}3\\-5\end{pmatrix} + \begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$6 + z_1 = 0 \Rightarrow z_1 = -6$

$-10 + z_2 = 0 \Rightarrow z_2 = 10$

$z = -6\mathbf{i} + 10\mathbf{j}$

**5** $\mathbf{a} = \mathbf{b} \Rightarrow \begin{pmatrix}x\\x + y\end{pmatrix} = \begin{pmatrix}6 - y\\-2x - 3\end{pmatrix}$

$x = 6 - y$ (1)

$x + y = -2x - 3$

$y = -3x - 3$ (2)

Sub (1) into (2)

$y = -3(6 - y) - 3$

$y = -18 + 3y - 3$

$2y = -21$

$y = \frac{-21}{2}$

$x = 6 - y = 6 - \left(\frac{-21}{2}\right) = \frac{33}{2}$

**6** $3\mathbf{a} = 2\mathbf{b} \Rightarrow 3\begin{pmatrix}3\\t\\u\end{pmatrix} = 2\begin{pmatrix}t - s\\3s\\t + s\end{pmatrix}$

(1) $9 = 2(t - s)$

(2) $3t = 6s$

(3) $3u = 2(t + s)$

(2) $\Rightarrow t = 2s$

(1) $\Rightarrow 9 = 2(2s - s)$

$= 2s$

$s = \frac{9}{2}$

(2) $\Rightarrow t = 9$

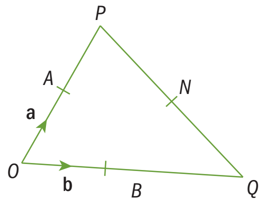
(3) $\Rightarrow 3u = 2\left(9 + \frac{9}{2}\right)$

$u = \frac{27}{3} = 9$

$t = 9, s = \frac{9}{2}, u = 9$

## Exercise 12H

**1**




**a** $\overrightarrow{AP} = \overrightarrow{OA} = \mathbf{a}$

**b** $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

**c** $\overrightarrow{PQ} = -\overrightarrow{AP} - \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BQ}$

$= -\mathbf{a} - \mathbf{a} + \mathbf{b} + 3\mathbf{b}$

$= 4\mathbf{b} - 2\mathbf{a}$

**d** $\overrightarrow{PN} = \frac{1}{2}\overrightarrow{PQ} = \frac{1}{2}(4\mathbf{b} - 2\mathbf{a})$

$= 2\mathbf{b} - \mathbf{a}$

**e** $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AP} + \overrightarrow{PN}$

$= \mathbf{a} + \mathbf{a} + (2\mathbf{b} - \mathbf{a})$

$= \mathbf{a} + 2\mathbf{b}$

**f** $\overrightarrow{AN} = \overrightarrow{AP} = \overrightarrow{PN}$

$= \mathbf{a} + (2\mathbf{b} - \mathbf{a})$

$= 2\mathbf{b}$

**2** $\mathbf{a} = \overrightarrow{OA}, \mathbf{b} = \overrightarrow{OB}, \overrightarrow{AC} : \overrightarrow{CB} = 3:1$

**a** $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

**b** $\overrightarrow{AC} = \frac{3}{4}\overrightarrow{AB}$

$= \frac{3}{4}(\mathbf{b} - \mathbf{a})$

**c** $\overrightarrow{CB} = \frac{1}{4}\overrightarrow{AB}$

$= \frac{1}{4}(\mathbf{b} - \mathbf{a})$

**d** $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \mathbf{a} + \frac{3}{4}(\mathbf{b} - \mathbf{a})$

$= \mathbf{a}\left(1 - \frac{3}{4}\right) + \mathbf{b}\frac{3}{4}$

$= \frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

**3** $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OC} = \mathbf{c}, \overrightarrow{CB} = 3\mathbf{a}$

**a** $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= \mathbf{c} + 3\mathbf{a}$

**b** $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{CB}$
$= -\mathbf{a} + \mathbf{c} + 3\mathbf{a}$
$= \mathbf{c} + 2\mathbf{a}$

**c** $\overrightarrow{CD} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
$= \mathbf{a} + \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} + \frac{1}{2}\mathbf{c}$

**d** $\overrightarrow{CD} = \overrightarrow{CB} - \frac{1}{2}\overrightarrow{AB}$
$= 3\mathbf{a} - \frac{1}{2}(\mathbf{c} + 2\mathbf{a})$
$= 2\mathbf{a} - \frac{1}{2}\mathbf{c}$

**4** $\overrightarrow{FA} = \mathbf{a}, \overrightarrow{FB} = \mathbf{b}$

**a** $\overrightarrow{AB} = -\overrightarrow{FA} + \overrightarrow{FB}$
$= -\mathbf{a} + \mathbf{b}$

**b** $\overrightarrow{FO} = \overrightarrow{FB} + \overrightarrow{BO}$
By symmetry, $\overrightarrow{OB} = \overrightarrow{FA} = \mathbf{a}$
so $\overrightarrow{FO} = \overrightarrow{FB} - \overrightarrow{OB}$
$= \mathbf{b} - \mathbf{a}$

**c** $\overrightarrow{FC} = \overrightarrow{FO} + \overrightarrow{OE} + \overrightarrow{EC}$
By symmetry, $\overrightarrow{OE} = \overrightarrow{BO} = -\mathbf{a}$
and $\overrightarrow{EC} = \overrightarrow{FB} = \mathbf{b}$
so $\overrightarrow{FC} = (\mathbf{b} - \mathbf{a}) - \mathbf{a} + \mathbf{b}$
$= 2(\mathbf{b} - \mathbf{a})$

**d** $\overrightarrow{BC} = \overrightarrow{BE} + \overrightarrow{EC}$
$= \overrightarrow{BO} + \overrightarrow{OC} + \overrightarrow{EC}$
$= -\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{EC}$
$= -\mathbf{a} - \mathbf{a} + \mathbf{b}$
$= \mathbf{b} - 2\mathbf{a}$

**e** $\overrightarrow{FD} = \overrightarrow{FC} + \overrightarrow{CD}$
By symmetry, $\overrightarrow{CD} = -\overrightarrow{FA} = -\mathbf{a}$
so $\overrightarrow{FD} = 2(\mathbf{b} - \mathbf{a}) - \mathbf{a} = 2\mathbf{b} - 3\mathbf{a}$

**5** $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$

**a** $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$
$= -\mathbf{a} + \mathbf{b}$
$= \mathbf{b} - \mathbf{a}$
since $\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AB}$
$\overrightarrow{AP} = \frac{2}{3}(\mathbf{b} - \mathbf{a})$

**b** M is mid point of OA, so
$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$
$\overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{AP}$
$= \frac{1}{2}\mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a})$
$= \left(\frac{1}{2} - \frac{2}{3}\right)\mathbf{a} + \frac{2}{3}\mathbf{b}$
$= \frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}$

**c** $\overrightarrow{MX} = \overrightarrow{MP} + \overrightarrow{PB} + \overrightarrow{BX}$
$\overrightarrow{PB} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$
$\overrightarrow{BX} = \overrightarrow{OB} = \mathbf{b}$
so $\overrightarrow{MX} = \left(\frac{2}{3}\mathbf{b} - \frac{1}{6}\mathbf{a}\right) + \frac{1}{3}(\mathbf{b} - \mathbf{a}) + \mathbf{b}$
$= 2\mathbf{b}$

## Exercise 12I

**1** **a** $\mathbf{a}\cdot\mathbf{b} = (2\mathbf{i} + 4\mathbf{j})\cdot(\mathbf{i} - 5\mathbf{j})$
$= (2\times1) + (4\times-5)$
$= 2 - 20$
$= -18$

**b** $\mathbf{b}\cdot\mathbf{c} = (\mathbf{i} - 5\mathbf{j})\cdot(-5\mathbf{i} - 2\mathbf{j})$
$= (1\times-5) + (-5\times-2)$
$= -5 + 10$
$= 5$

**c** $\mathbf{a}\cdot\mathbf{a} = (2\mathbf{i} + 4\mathbf{j})\cdot(2\mathbf{i} + 4\mathbf{j})$
$= (2\times2) + (4\times4)$
$= 4 + 16$
$= 20$

**d** $\mathbf{c}\cdot(\mathbf{a} + \mathbf{b}) = (-5\mathbf{i} - 2\mathbf{j})\cdot[(2\mathbf{i} + 4\mathbf{j}) + (-\mathbf{i} - 5\mathbf{j})]$
$= (-5\mathbf{i} - 2\mathbf{j})\cdot[\mathbf{i} - \mathbf{j}]$
$= (-5\times1) + (-2\times-1)$
$= -5 + 2$
$= -3$

**e** $(\mathbf{c} + \mathbf{a})\cdot\mathbf{b} = [(-5\mathbf{i} - 2\mathbf{j}) + (2\mathbf{i} + 4\mathbf{j})]\cdot(\mathbf{i} - 5\mathbf{j})$
$= [-3\mathbf{i} + 2\mathbf{j}]\cdot(\mathbf{i} - 5\mathbf{j})$
$= (-3\times1) + (2\times-5)$
$= -3 - 10 = -13$

**2** **a** $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4\\-3\\-1\end{pmatrix} = (-1\times4) + (0\times-3) + (5\times-1)$
$= -4 + 0 - 5$
$= -9$

**b** $\mathbf{u}\cdot(\mathbf{v} - \mathbf{w}) = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}4 - (-1)\\-3 - (3)\\-1 - (-6)\end{pmatrix} = \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\5\end{pmatrix}$
$= (-1\times5) + (0\times-6) + (5\times5)$
$= -5 + 0 + 25$
$= 20$

**c** $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = -9 - \begin{pmatrix}-1\\0\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-6\end{pmatrix}$
$= -9 - [(-1)\times(-1) + 0\times3 + 5\times(-6)]$
$= -9 - [1 + 0 - 30]$
$= -9 + 29 = 20$

**d** $2\,\mathbf{u}\cdot\mathbf{w} = 2(-29)$
$= -58$

**e** $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{w}) = \begin{bmatrix} -1 & -4 \\ 0 & -(-3) \\ 5 & -(-1) \end{bmatrix}\cdot\begin{bmatrix} -1 & +(-1) \\ 0 & +3 \\ 5 & +(-6) \end{bmatrix}$
$= \begin{pmatrix} -5 \\ 3 \\ 6 \end{pmatrix}\cdot\begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$
$= 10 + 9 - 6 = 13$

**3** **a** $\mathbf{a}\cdot\mathbf{b} = (2\times 4) + (4\times -2)$
$= 8 - 8$
$= 0 \Rightarrow$ perpendicular.

**b** $\mathbf{c}\cdot\mathbf{d} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (2\times 1) + (1\times 2) = 4$
$|\mathbf{c}| = \sqrt{2^2 + 1^2} = \sqrt{5} \quad |\mathbf{d}| = \sqrt{5}$
$|\mathbf{c}|\,|\mathbf{d}| = \left(\sqrt{5}\right)^2 = 5 \neq \mathbf{c}\cdot\mathbf{d}$

So neither parallel, nor perpendicular.

**c** $\mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} -8 \\ 2 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} = (-8\times 4) + (2\times -1) + (2\times -1)$
$= -32 - 2 - 2 = -36$
$|\mathbf{u}| = \sqrt{8^2 + 2^2 + 2^2} = \sqrt{64+4+4} = \sqrt{72}$
$|\mathbf{v}| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{16+1+1} = \sqrt{18}$
$|\mathbf{u}||\mathbf{v}| = \sqrt{18\times 72} = 36 = -\mathbf{u}\cdot\mathbf{v}$
$\Rightarrow$ parallel.

**d** $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\mathbf{a}\cdot\mathbf{b} = (3\times 3) + (-2\times -2) + (1\times -1)$
$= 9 + 4 - 1$
$= 12$
$|\mathbf{a}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$
$|\mathbf{b}| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$
$|\mathbf{a}||\mathbf{b}| = \left(\sqrt{14}\right)^2 = 14 \neq \mathbf{a}\cdot\mathbf{b}$
$\Rightarrow$ neither parallel, nor perpendicular

**e** $\overrightarrow{OX}\cdot\overrightarrow{OZ} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = (1\times 0) + (0\times 0) + (0\times 1) = 0$
$\Rightarrow$ perpendicular

**f** $\mathbf{n}\cdot\mathbf{m} = (2\mathbf{i} - 8\mathbf{j})(-\mathbf{i} + 4\mathbf{j})$
$= (2\times -1) + (-8\times 4) = -2 - 32 = -34$
$|\mathbf{n}| = \sqrt{2^2 + 8^2} = \sqrt{4+64} = \sqrt{68}$
$|\mathbf{m}| = \sqrt{1^2 + 4^2} = \sqrt{1+16} = \sqrt{17}$
$|\mathbf{n}||\mathbf{m}| = \sqrt{17\times 68} = 34 = -\mathbf{n}\cdot\mathbf{m}$
$\Rightarrow$ parallel.

**g** $\overrightarrow{AB}\cdot\overrightarrow{CD} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ -1 \end{pmatrix} = (2\times -1) + (2\times -1)$
$= -2 - 2$
$= -4$
$|\overrightarrow{AB}| = \sqrt{2^2 + 2^2} = \sqrt{8}$
$|\overrightarrow{CD}| = \sqrt{1^2 + 1^2} = \sqrt{2}$
$|\overrightarrow{AB}|\cdot\overrightarrow{CD} = \sqrt{2\times 8} = \sqrt{16} = 4 = -\,\overrightarrow{AB}\cdot\overrightarrow{CD}$
$\Rightarrow$ parallel vectors

**4** $\mathbf{a} + 3\mathbf{b} = (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) + 3(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= (1 + 3\times 3)\mathbf{i} + (1 + 3\times 2)\mathbf{j} + (2 + 3\times -1)\mathbf{k}$
$= 10\mathbf{i} + 7\mathbf{j} - \mathbf{k}$
$2\mathbf{a} - \mathbf{b} = 2(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= (2(1) - 3)\mathbf{i} + (2(1) - 2)\mathbf{j} + (2(2) - (-1))\mathbf{k}$
$= -\mathbf{i} + 5\mathbf{k}$
$(\mathbf{a} + 3\mathbf{b})\cdot(2\mathbf{a} - \mathbf{b}) = (10\mathbf{i} + 7\mathbf{j} - \mathbf{k})\cdot(\mathbf{i} + 5\mathbf{k})$
$= (10\times 1) + (-1\times 5)$
$= 10 - 5 = 5$

**5** Let $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$
$\mathbf{a}\cdot\mathbf{d} = 3d_1 + (-5)d_3 = -9$
$\mathbf{b}\cdot\mathbf{d} = 2d_1 + 7d_2 = 11$
$\mathbf{c}\cdot\mathbf{d} = d_1 + d_2 + d_3 = 6$
using GDC, $d_1 = 2$, $d_2 = 1$, $d_3 = 3$

So, $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

**6** $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cos\theta$
$\sqrt{6} = 2\sqrt{3}\cos\theta$
$\cos\theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$

**7** **a** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\times 2 + -1\times 5 = 4 - 5 = -1$
$\left|\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + 1^2} = \sqrt{5}$
$\left|\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right| = \sqrt{2^2 + 5^2} = \sqrt{4+25} = \sqrt{29}$
$-1 = \sqrt{5}\sqrt{29}\cos\theta$
$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{145}}\right) = 94.8°$

**b** $\begin{pmatrix} 4 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} -3 \\ 1 \end{pmatrix} = 4\times -3 + 0\times 1 = -12$
$\left|\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right| = \sqrt{4^2} = 4$
$\left|\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right| = \sqrt{3^2 + 1^2} = \sqrt{10}$

$-12 = 4\sqrt{10}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) = 161.6°$

**c** $(2\mathbf{i} + 5\mathbf{j}) \cdot (2\mathbf{i} - 5\mathbf{j}) = (2 \times 2) + (5 \times -5)$

$= 4 - 25$

$= -21$

$|(2\mathbf{i} + 5\mathbf{j})| = \sqrt{2^2 + 5^2} = \sqrt{29} = |(2\mathbf{i} - 5\mathbf{j})|$

$\therefore -21 = 29\cos\theta$

$\theta = \cos^{-1}\left(\frac{-21}{29}\right) = 136.4°$

**8 a** $\overrightarrow{AB} = \begin{pmatrix}1\\9\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-1\\5\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

**b** $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix}-1\\5\end{pmatrix} \cdot \begin{pmatrix}1\\-2\end{pmatrix} = (-1 \times 1) + (5 \times -2)$

$= -1 - 10 = -11$

**c** $|\overrightarrow{AB}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$|\overrightarrow{AC}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\cos\theta$

$-11 = \sqrt{5 \times 26}\,\cos\theta$

$\cos\theta = \frac{-11}{\sqrt{130}}$

**9 a** $\begin{pmatrix}-1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\6\end{pmatrix} = -2 - 6 + 12 = 4$

$\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

$\left|\begin{pmatrix}2\\-3\\6\end{pmatrix}\right| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = 7$

so $4 = 3 \times 7\cos\theta$

$\cos\theta = \frac{4}{21}, \theta = 79°$

**b** $\begin{pmatrix}2\\3\\1\end{pmatrix} \cdot \begin{pmatrix}4\\-2\\-2\end{pmatrix} = 8 - 6 - 2 = 0 \Rightarrow$ perpendicular vectors

$\theta = 90°$

**c** $(2\mathbf{i} - 7\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - \mathbf{k})$

$= (2 \times 1) + (-7 \times 1) + (1 \times -1) = -6$

$|(2\mathbf{i} - 7\mathbf{j} + \mathbf{k})| = \sqrt{2^2 + 7^2 + 1^2} = \sqrt{54}$

$|(\mathbf{i} + \mathbf{j} - \mathbf{k})| = \sqrt{1 + 1 + 1} = \sqrt{3}$

so $-6 = \sqrt{162}\cos\theta$

$\theta = \cos^{-1}\left(\frac{-6}{\sqrt{162}}\right) = 118.1°$

**10 a** $\overrightarrow{AB} = \begin{pmatrix}2\\3\\4\end{pmatrix} - \begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}1\\4\\0\end{pmatrix}$

$\overrightarrow{AB} = = \sqrt{1^2 + 4^2} = \sqrt{17}$

$\overrightarrow{AC} = \begin{pmatrix}2\\-1\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}1\\0\\-5\end{pmatrix}$

$\overrightarrow{AC} = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

so $AB = \sqrt{17}$, $AC = \sqrt{26}$

**b** $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix}1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-5\end{pmatrix} = 1$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}|\cos\theta$

$1 = \sqrt{17}\sqrt{26}\cos\theta$

$\frac{1}{\sqrt{442}} = \cos\theta$

**c** Area $ABC = \frac{1}{2}\,|\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin B\hat{A}C$

$= \frac{1}{2}\sqrt{442}\sin\left(\cos^{-1}\frac{1}{\sqrt{442}}\right) = 10.5\text{cm}^2$

**11** The $x$-axis has unit direction vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$

so $\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

$\left|\begin{pmatrix}1\\0\\0\end{pmatrix}\right| = \sqrt{1^2} = 1$ and $\left|\begin{pmatrix}1\\1\\1\end{pmatrix}\right| = \sqrt{1 + 1 + 1} = \sqrt{3}$

so $1 = \sqrt{3}\cos\theta$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54.7°$

**12 a** $\overrightarrow{OA} = 4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$(\overrightarrow{OA}) . (\overrightarrow{OB}) = (4 \times 1) + (4 \times 2) + (-4 \times 3)$

$= 4 + 8 - 12 = 0$

$=$ $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are perpendicular

**b** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (1 - 4)\mathbf{i} + (2 - 4)\mathbf{j} + (3 - (-4))\mathbf{k}$

$= 3\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$

$|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-2)^2 + 7^2} = \sqrt{62} \approx 7.87$

**13** $(2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = (2 \times 1) + (\lambda \times -2) + (3 \times 1)$

$= 2 - 2\lambda + 3 = 0$

for perpendicular vectors.

$5 = 2\lambda$

$\lambda = \frac{5}{2}$

**14** $\mathbf{a} + \mathbf{b} = (5 + 1)\mathbf{i} + (-3 + 1)\mathbf{j} + (7 + \lambda)\mathbf{k}$

$= 6\mathbf{i} - 2\mathbf{j} + (7 + \lambda)\mathbf{k}$

$\mathbf{a} - \mathbf{b} = (5 - 1)\mathbf{i} + (-3 - 1)\mathbf{j} + (7 - \lambda)\mathbf{k}$

$= 4\mathbf{i} - 4\mathbf{j} + (7 - \lambda)\mathbf{k}$

Now $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (6 \times 4) + (-2 \times -4) + (\lambda + 7)(7 - \lambda)$

$= 24 + 8 + 49 - \lambda^2 = 0$

$\lambda^2 = 81$

$\lambda^2 = \pm 9$

**15** $\mathbf{a}+\mathbf{b}=\begin{pmatrix}p\\2\\-p\end{pmatrix}+\begin{pmatrix}2\\-p\\-3\end{pmatrix}=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}$

$\mathbf{a}-\mathbf{b}=\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\begin{pmatrix}p+2\\2-p\\-p-3\end{pmatrix}\cdot\begin{pmatrix}p-2\\2+p\\-p+3\end{pmatrix}$

$=(p^2-4)+(4-p^2)-(9-p^2)$

$= p^2 - 9 = 0$ for perpendicular vectors.

$\Rightarrow p^2 = 9,\ p = \pm 3$

## Exercise 12J

**1** **a** $\mathbf{r}=\begin{pmatrix}-1\\2\end{pmatrix}+t\begin{pmatrix}3\\2\end{pmatrix},\ t\in\mathbb{R}.$

**b** $\mathbf{r}=\begin{pmatrix}-1\\0\end{pmatrix}+t\begin{pmatrix}5\\-2\end{pmatrix},\ t\in\mathbb{R}.$

**c** $\mathbf{r}=\begin{pmatrix}3\\1\\-2\end{pmatrix}+t\begin{pmatrix}3\\-2\\8\end{pmatrix},\ t\in\mathbb{R}.$

**d** $\mathbf{r}=2\mathbf{j}+\mathbf{k}+t(3\mathbf{i}-\mathbf{j}+\mathbf{k}),\ t\in\mathbb{R}.$

**2** **a** Position vectors are $\begin{pmatrix}4\\5\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$

Line joining the 2 points has direction

$\begin{pmatrix}3 & -4\\-2 & -5\end{pmatrix}=\begin{pmatrix}-1\\-7\end{pmatrix}$

Line is $\mathbf{r}=\begin{pmatrix}4\\5\end{pmatrix}+t\begin{pmatrix}1\\7\end{pmatrix},\ t\in\mathbb{R}.$

**b** Position vectors $\begin{pmatrix}4\\-2\end{pmatrix}$ and $\begin{pmatrix}5\\-2\end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix}5 & -4\\-2 & (-2)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$

Line is $\mathbf{r}=\begin{pmatrix}4\\-2\end{pmatrix}+t\begin{pmatrix}1\\0\end{pmatrix},\ t\in\mathbb{R}.$

**c** Position vectors $\begin{pmatrix}3\\5\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-4\\5\end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix}3 & -2\\5 & -(-4)\\2 & -5\end{pmatrix}=\begin{pmatrix}1\\9\\-3\end{pmatrix}$

Line is $\mathbf{r}=\begin{pmatrix}3\\5\\2\end{pmatrix}+t\begin{pmatrix}1\\9\\3\end{pmatrix},\ t\in\mathbb{R}.$

**d** Position vectors $\begin{pmatrix}0\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

Line joining 2 points has direction

$\begin{pmatrix}0 & -1\\0 & -(-1)\\1 & -0\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$

Line is $\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}$

**3** **a** We need a vector $\mathbf{p}=\begin{pmatrix}p_1\\p_2\end{pmatrix}$ which is perpendicular to $\mathbf{a}$

$\mathbf{a}\cdot\mathbf{p}=0\Rightarrow\begin{pmatrix}3\\2\end{pmatrix}\cdot\begin{pmatrix}p_1\\p_2\end{pmatrix}=3p_1+2p_2=0$

Take $p_1 = 2,\ p_2 = -3$

Then $\mathbf{a}\cdot\mathbf{p} = 0$, and line is

$\mathbf{r}=\begin{pmatrix}-1\\6\end{pmatrix}+t\begin{pmatrix}2\\-3\end{pmatrix},\ t\in\mathbb{R}.$

**b** Using the same technique as in part **a**, we see

$\begin{pmatrix}2\\5\end{pmatrix}$ is perpendicular $\begin{pmatrix}5\\-2\end{pmatrix}$

Line is $\mathbf{r}=\begin{pmatrix}-1\\0\end{pmatrix}+t\begin{pmatrix}2\\5\end{pmatrix},\ t\in\mathbb{R}.$

**c** $\begin{pmatrix}1\\0\\3\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\0\\-1\end{pmatrix}$

line is $\mathbf{r} = \begin{pmatrix}4\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\0\\3\end{pmatrix}, t \in \mathbb{R}$.

**d** We require $\mathbf{p} = \begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}$ so that $\mathbf{p}\cdot\mathbf{a} = 0$

$\mathbf{p}\cdot\mathbf{a} = p_1 - 3p_2 + 4p_3 = 0$

Take $p_1 = 0, p_2 = 4, p_3 = 3$

Then line is $\mathbf{r} = 5\mathbf{k} + t(4\mathbf{j} + 3\mathbf{k}), t \in \mathbb{R}$.

**4** **a** We need to know if there is a value of $t$ for which

$\mathbf{r} = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}4\\5\end{pmatrix}$

Take $t = 2$ Then $\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$

so (4, 5) lies on the line.

**b** Is there $t$ so that $\begin{pmatrix}5\\1\end{pmatrix} + t\begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}5\\-2\end{pmatrix}$?

$5 + t(4) = 5$ and $1 - 3t = -2$

$t = 0$ and $t = 1 \Rightarrow$ no such $t$.

so (5, –2) does not lie on the line.

**c** Is there $t$ so that $\begin{pmatrix}-1\\5\\-3\end{pmatrix} + t\begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}-3\\5\\1\end{pmatrix}$?

$-1 + t = -3 \Rightarrow t = -2$

$5 + 0(t) = 5 \Rightarrow t =$ anything

$-3 - 2t = 1 \Rightarrow t = -2$

(–3, 5, 1) so $\begin{pmatrix}-3\\5\\1\end{pmatrix} = \begin{pmatrix}-1\\5\\-3\end{pmatrix} - 2\begin{pmatrix}1\\0\\-2\end{pmatrix}$ i.e. lies on line.

**d** Is there $t$ so that

$(2\mathbf{i} + \mathbf{j} + \mathbf{k}) = (2\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + t(-2\mathbf{j} - 3\mathbf{k})$

$1 = -1 - 2t$ and $1 = -3 - 3t$

$-2 = 2t \qquad 4 = -3t$

$t = -1$ and $t = \frac{-4}{3} \Rightarrow$ no such $t$.

so (2,1,1) does not lie on line.

**5** $\mathbf{r} = \begin{pmatrix}2\\4\\5\end{pmatrix} + t\begin{pmatrix}-2\\3\\8\end{pmatrix}, t \in \mathbb{R}$

$10 = 4 + 3t \Rightarrow 6 = 3t, t = 2$

$P = 2 - 2t = 2 - 2(2) = -2$

$q = 5 + 8t = 5 + 8(2) = 21$

**6** A vertical line will have direction $\begin{pmatrix}0\\1\end{pmatrix}$

so $\mathbf{r} = \begin{pmatrix}-6\\5\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}, t \in \mathbb{R}$

**7** **a** (1) Are the 2 lines parallel?

Is there $t$ such that $\begin{pmatrix}2\\-1\end{pmatrix} = t\begin{pmatrix}-6\\3\end{pmatrix}$

Take $t = \frac{-1}{3}$. Then $\begin{pmatrix}2\\-1\end{pmatrix} = \frac{-1}{3}\begin{pmatrix}-6\\3\end{pmatrix}$ so lines parallel.

(2) Are 2 lines co-incident?

Does $\begin{pmatrix}-9\\10\end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix}-9\\10\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} + s\begin{pmatrix}2\\-1\end{pmatrix}$. Take $s = -6$.

$\begin{pmatrix}-9\\10\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} - 6\begin{pmatrix}2\\-1\end{pmatrix}$

so $\begin{pmatrix}-9\\10\end{pmatrix}$ lies on $\mathbf{r}_1 \Rightarrow$ lines co-incident

**b** (1) Are lines parallel?

Is there $t$ so that $\begin{pmatrix}-4\\2\end{pmatrix} = t\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow$ No such t, so NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix}-4\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix} = -4 + 4 = 0 \Rightarrow$ perpendicular.

**c** (1) Are the lines parallel?

Is there $t$ so that $\begin{pmatrix}4\\-3\end{pmatrix} = t\begin{pmatrix}8\\-6\end{pmatrix}$

$t = 2$ gives

$\begin{pmatrix}4\\-3\end{pmatrix} = 2\begin{pmatrix}8\\-6\end{pmatrix} \Rightarrow$ lines parallel.

(2) Are lines co-incident?

Does $\begin{pmatrix}5\\3\end{pmatrix}$ lie on $\mathbf{r}_1$?

$\begin{pmatrix}5\\3\end{pmatrix} = \begin{pmatrix}5\\1\end{pmatrix} + s\begin{pmatrix}4\\-3\end{pmatrix}$ No such $s \Rightarrow$

lines NOT co-incident.

**d** (1) Are lines parallel?

Is there $t$ so that $\begin{pmatrix}1\\2\end{pmatrix} = t\begin{pmatrix}1\\1\end{pmatrix}$ No such $t$

$\Rightarrow$ NOT parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = 1 + 2 = 3 \Rightarrow$ NOT perpendicular.

**e** (1) Are lines parallel?

Is there $t$ so that

$\begin{pmatrix}4\\-3\end{pmatrix} = t\begin{pmatrix}4\\3\end{pmatrix}$ No such $t \Rightarrow$

lines not parallel.

(2) Are lines perpendicular?

Take dot product of direction vectors:

$\begin{pmatrix}4\\-3\end{pmatrix}.\begin{pmatrix}4\\3\end{pmatrix} = 16 - 9 = 7 \Rightarrow$

NOT perpendicular.

**9** **a** A has position vector $\begin{pmatrix}-2\\-3\\-4\end{pmatrix}$. We require $t$ so that

$\begin{pmatrix}-2\\-3\\-4\end{pmatrix} = \begin{pmatrix}-1\\-1\\2\end{pmatrix} + t\begin{pmatrix}1\\2\\6\end{pmatrix}$ Taking $t = -1$, we see:

$-2 = -1 + (-1)1$

$-3 = -1 + (-1)2$

$-4 = 2 + (-1)\,6$

**b** $\overrightarrow{AB} = \begin{pmatrix}-6\\-7\\-2\end{pmatrix} - \begin{pmatrix}-2\\-3\\-4\end{pmatrix} = \begin{pmatrix}-4\\-4\\2\end{pmatrix}$

Taking dot product,

$\begin{pmatrix}-4\\-4\\2\end{pmatrix}.\begin{pmatrix}1\\2\\6\end{pmatrix} = -4 - 8 + 12 = 0$

$\Rightarrow \overrightarrow{AB}$ perpendicular to $\ell 1$

**10** **a** **i** $\overrightarrow{OF} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

**ii** $\overrightarrow{AG} = \mathbf{j} + \mathbf{k}$

**b** **i** $|\overrightarrow{OF}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

**ii** $|\overrightarrow{AG}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

**iii** $\overrightarrow{OF} \,.\, \overrightarrow{AG} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \,.\, (\mathbf{j} + \mathbf{k}) = 2$

**c** $\overrightarrow{OF} \,.\, \overrightarrow{AG} = |\overrightarrow{OF}|\;|\overrightarrow{AG}|\cos\theta$

$2 = \sqrt{2}\sqrt{3}\cos\theta$

$\theta = \cos\theta^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.3^\circ$

**11** **a** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$

$= 4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$

**b** $(OB)^2 = (OA)^2 + (AB)^2 - 2(OA)(AB)\cos O\hat{A}B$

$(OB)^2 = 5^2 + 3^2 + 6^2 = 70, (OA)^2 = 1^2 + 5^2 + 2^2 = 30$

$(AB)^2 = 4^2 + 8^2 + 8^2 = 144$

$O\hat{A}B = \cos^{-1}\left[\frac{-(70 - 30 - 144)}{2 \times 30 \times 144}\right] \approx 89.7^\circ$

**c** Line through A & B is given by

$\mathbf{r} = \begin{pmatrix}1\\5\\-2\end{pmatrix} + t\begin{pmatrix}4\\-8\\8\end{pmatrix}$

Suppose $\mu \in \mathbb{R}$. Then we need $t$ so that

$1 + 4t = \mu$ and $5 - 8t = 7 - 2\mu$

$t = \frac{1}{4}(\mu - 1)$ $\quad 2\mu = 2 + 8t$

$t = \frac{1}{4}(\mu - 1)$

and $-2 + 8t = 2\mu$

$8t = 2\mu + 2$

$t = \frac{1}{4}(\mu + 1)$

$t$ is the same value for each component, so $\mu\mathbf{i} + (7 - 2\mu)\mathbf{j} + 2\mu\mathbf{k}$ lies on line through A & B.

**d** $\begin{pmatrix}\mu\\7-2\mu\\2\mu\end{pmatrix}.\begin{pmatrix}4\\-8\\8\end{pmatrix} = 4\mu - 8(7 - 2\mu) + 16\mu = 0$

$4\mu - 56 + 16\mu + 16\mu = 0$

$36\mu - 56 = 0$

$\mu = \frac{56}{36} = \frac{14}{9}$

**e** $\frac{14}{9}\mathbf{i} + \left(7 - 2\left(\frac{14}{9}\right)\right)\mathbf{j} + 2\left(\frac{14}{9}\right)\mathbf{k}$

$= \frac{14}{9}\mathbf{i} + \frac{35}{9}\mathbf{j} + \frac{28}{9}\mathbf{k}$

## Exercise 12K

**1** Equating components of $\mathbf{r}_1$ & $\mathbf{r}_2$:

$4 + 2\lambda = 11 + \mu$ (1)

$2 - 4\lambda = 16 + 2\mu$ (2)

(1) $\Rightarrow \mu = -7 + 2\lambda$

(2) $\Rightarrow 2 - 4\lambda = 16 + 2(-7 + 2\lambda)$

$2 - 4\lambda = 2 + 2\lambda$

$\lambda = 0$

(1) $\Rightarrow \mu = -7$

so intercept at (4,2)

**2** Equating components:

$4 + 8s = 6 + 9t$ (1)

$-2 + 2s = -3 + 6t$ (2)

(1) $\Rightarrow 8s = 2 + 9t$

$s = \frac{1}{8}(2 + 9t)$

(2) $\Rightarrow -2 + \frac{1}{4}(2 + 9t) = -3 + 6t$

$-8 + 2 + 9t = -12 + 24t$

$6 = 15t$

$t = \frac{6}{15}$

$(1) \Rightarrow s = \frac{1}{8}\left(2 + \frac{54}{15}\right) = \frac{7}{10}$

intercept at $\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \frac{7}{10}\begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{48}{5} \\ \frac{-3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 48 \\ -3 \end{pmatrix}$

point $\left(\frac{48}{5}, \frac{-3}{5}\right)$

**3** $5 + 2t = 3 + 2s \quad (1)$

$-1 + t = -2 + s \quad (2)$

$2 - t = -4 + 2s \quad (3)$

$(2) \Rightarrow s = 1 + t$

$(1) \Rightarrow 5 + 2t = 3 + 2(1 + t)$

$5 + 2t = 5 + 2t$ (so (1) & (2) are consistent)

$(3)\ 2 - t = -4 + 2(1 + t)$

$2 - t = -2 + 2t$

$4 = 3t$

$t = \frac{4}{3}$

$(2) \Rightarrow s = 1 + \frac{4}{3} = \frac{7}{3}$

Thus $\ell_1$ & $\ell_2$ intersect.

$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \frac{7}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 23 \\ 1 \\ 2 \end{pmatrix}$ at $\left(\frac{23}{3}, \frac{1}{3}, \frac{2}{3}\right)$

**4** $1 + 3t = -1 \quad (1)$

$1 - t = s \quad (2)$

$(1) \Rightarrow 3t = -2$

$t = \frac{-2}{3}$

$(2) \Rightarrow s = 1 - \left(\frac{-2}{3}\right) = \frac{5}{3}$

Intersect at $\mathbf{i} + \frac{5}{3}\mathbf{j}$, i.e. at $\left(-1, \frac{5}{3}\right)$

**5** If lines Intersect then they are $s$ & $t$ so that

$3 - t = 1 + s \quad (1)$

$t = 4 + s \quad (2)$

$5 + 2t = s \quad (3)$

sub (2) into (3): $s = 5 + 2(4 + s)$, $s = 13 + 2s$

$s = -13$

in (2) $\Rightarrow t = 4 - 13 = -9$

check in (1): $3 - (-9) \neq 1 - 13$

$12 \neq -12$

so there are no such $s$ & $t \Rightarrow$ skew

**6 a** $3 - s = 14 + 3t \quad (1)$

$-2 - 3s = -20 - 4t \quad (2)$

$5 - 5s = 6 - 3t \quad (3)$

$(1) \Rightarrow s = -11 - 3t$

$(2) \Rightarrow -2 - 3(11 + 3t) = -20 - 4t$

$-35 - 9t = -20 - 4t$

$-15 = 5t$

$t = -3$

$(1) \Rightarrow s = -11 - 3(-3) = -2$

check in (3): $5 - 5(-2) = 6 - 3(-3)$

$15 = 15$ so lines intersect.

**b** Take dot product of direction vectors:

$(-\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}) \cdot (3\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

$= (-1 \times 3) + (3 \times -4) + (-5 \times -3)$

$= -3 - 12 + 15$

$= 0$

$\Rightarrow$ perpendicular.

**7 a** $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ a \end{pmatrix}$

$6 + t = 5 \Rightarrow t = -1$

$3 - 2(-1) = a \Rightarrow a = 5.$

$\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} b \\ 13 \\ -1 \end{pmatrix}$

$9 + 2t = 13 \Rightarrow t = 2.$

$6 + 2 = b \Rightarrow b = 8.$

**b** $OP$ has position vector $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ for some $t$

$\overrightarrow{AB} = \begin{pmatrix} 8 \\ 13 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix}$

$(\overrightarrow{OP}) \cdot (\overrightarrow{AB}) = 0 \Rightarrow \begin{pmatrix} 6+t \\ 9+2t \\ 3-2t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \\ -6 \end{pmatrix} = 0$

$3(6 + t) + 6(9 + 2t) - 6(3 - 2t) = 0$

$18 + 3t + 54 + 12t - 18 + 12t = 0$

$27t + 54 = 0$

$t = 2$

so $\overrightarrow{OP} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \\ -13 \end{pmatrix}$, P is

$(8, 13, -13)$

**c** $|\overrightarrow{OP}| = \sqrt{8^2 + 13^2 + 3^2} = \sqrt{242}$

**8** **a** $\overrightarrow{ab} = \mathbf{b} - \mathbf{a} = (3-2)\mathbf{i} + (-2-(-1))\,\mathbf{j} + (-1-2)\mathbf{k}$
$= \mathbf{i} - \mathbf{j} - 3\mathbf{k}$

line is $(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - 3\mathbf{k})$ for $\lambda \in \mathbb{R}$

**b** $2 + \lambda = 7 + 2s$ (1)
$-1 - \lambda = s$ (2)
$2 - 3\lambda = 3 + 2s$ (3)
sub (2) in (1) $\Rightarrow 2 + \lambda = 7 + 2(-1 - \lambda)$
$2 + \lambda = 5 - 2\lambda$
$3\lambda = 3 \Rightarrow \lambda = 1$
(2) $\Rightarrow s = -1 - 1 = -2$
(3) $\Rightarrow -3(1) = 3 + 2(-2)$
$-1 = -1$ lines intersect.
point is $(2+1)\mathbf{i} + (-1-1)\mathbf{j} + (2-3)\mathbf{k}$
$3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ ie $(3, -2, -1)$

**c** $\mathbf{a} - \mathbf{c} = (2-3)\mathbf{i} + (-1-(-2))\,\mathbf{j} + (2-(-1))\mathbf{k}$
$= -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$|AC| = \sqrt{1+1+3^2} = \sqrt{11}$

**d** Take dot product of direction vectors:
$(\mathbf{i} - \mathbf{j} - 3\mathbf{k}).(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 2 - 1 - 6 = -5$
Then $-5 = \sqrt{11}\,\sqrt{9}\cos\theta$
$\theta = \cos^{-1}\left(\frac{-5}{3\sqrt{11}}\right) = -60^\circ$ (nearest degree)

## Exercise 12L

**1** **a** Position of ship relative to buoy is
$\begin{pmatrix} 60 \\ 30 \end{pmatrix} - \begin{pmatrix} 45 \\ 20 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \end{pmatrix}$ ie 10Km North, 15Km East

**b** $\left|\begin{pmatrix} 15 \\ 20 \end{pmatrix}\right| = \sqrt{15^2 + 20^2} = 25$

**2** **a** velocity $= \frac{\text{displacement}}{\text{time}} = \begin{pmatrix} \frac{20}{4} \\ \frac{-8}{4} \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}\text{ms}^{-1}$

**b** $\mathbf{s}(t) = \begin{pmatrix} 20 \\ -8 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \end{pmatrix}$
$= \begin{pmatrix} 20 \\ -8 \end{pmatrix} + 6\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 50 \\ -20 \end{pmatrix}\text{m}$

**c** speed $= |\text{v(t)}| = \sqrt{12^2 + 5^2} = 13$
$\mathbf{s}(t) = (4\mathbf{i} - \mathbf{j}) + t(12\mathbf{i} - 5\mathbf{j})$
$\mathbf{s}(3) = (4\mathbf{i} - \mathbf{j}) + 3(12\mathbf{i} - 5\mathbf{j}) = 40\mathbf{i} - 16\mathbf{j}$

want $\begin{pmatrix} 20 \\ -8 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 12 \\ -5 \end{pmatrix}$
for collision
$10 + 5t = 4 + 12s$ (1)
$-8 - 2t = -1 - 5s$ (2)
(2) $\Rightarrow -7 + 5s = 2t \Rightarrow t = \frac{1}{2}(5s - 7)$
(1) $\Rightarrow 10 + \frac{5}{2}(5s - 7) = 4 + 12s$

$20 + 25s - 35 = 8 + 24s$
$s = 23$
(2) $\Rightarrow t = \frac{1}{2}(5 \times 23 - 7) = 54$
times not the same $\Rightarrow$ won't collide.

**3** **a** Let A's position be given by **a**
$\mathbf{a} = (3\mathbf{i} + 3\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$
Let B's position be given by **b**.
$\mathbf{b} = (4\mathbf{i} + 3\mathbf{j}) + s(3\mathbf{i} + 3\mathbf{j})$
want to find when **a** = **b**.
$3 + 4t = 4 + 3s$ (1)
$3 + 3t = 3 + 3s$ (2)
(2) $\Rightarrow t = s$
(1) $\Rightarrow s = t = 1$
They collide 1 hour after 3pm, ie 4pm.

**b** Collide at $(3\mathbf{i} + 3\mathbf{j}) + 1(4\mathbf{i} + 3\mathbf{j}) = 7\mathbf{i} + 6\mathbf{j}$

**4** **a** $\mathbf{r}_x = \begin{pmatrix} 11 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ $\mathbf{r}_y = \begin{pmatrix} 1 \\ -7 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$

$|\mathbf{V}_x| = \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$
$|\mathbf{V}_y| = \sqrt{2^2 + 1^2 + 9^2} = \sqrt{86}$

**b** Meet if $\mathbf{r}_x = \mathbf{r}_y$ at the same time
$\begin{pmatrix} 11 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ -2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix}$
$11 + t = 1 + 2s$ (1)
$3 - t = -7 + s$ (2)
$-3 + 4t = -2 + 9s$ (3)
(1) $\Rightarrow t - 2s - 10$
(2) $\Rightarrow 3 - (2s - 10) = -7 + s$
$13 - 2s = -7 + s$

$20 = 3s,\ s = \frac{20}{3}$
(1) $\Rightarrow t = 2\left(\frac{20}{3}\right) - 10 = \frac{10}{3} \neq s$
so ships do not collide.

**c** $\mathbf{r}_x(10) = \begin{pmatrix}11\\3\\-3\end{pmatrix} + 10\begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}21\\-7\\37\end{pmatrix}$

$\mathbf{r}_y(10) = \begin{pmatrix}1\\-7\\-2\end{pmatrix} + 10\begin{pmatrix}2\\1\\9\end{pmatrix} = \begin{pmatrix}21\\3\\88\end{pmatrix}$

$\mathbf{r}_y - \mathbf{r}_x = \begin{pmatrix}0\\10\\51\end{pmatrix},$

$|\mathbf{r}_y - \mathbf{r}_x|$

$= \sqrt{10^2 + 51^2} = \sqrt{2701} \approx 51.97\text{m}$

## Review exercise

**1** $\overrightarrow{AB} = \begin{pmatrix}-2\\3\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}7\\0\\-1\end{pmatrix} - \begin{pmatrix}-2\\3\\5\end{pmatrix} = \begin{pmatrix}9\\-3\\-6\end{pmatrix}$

Now $\overrightarrow{AB} = \begin{pmatrix}-3\\1\\2\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}9\\-3\\-6\end{pmatrix} = \frac{-1}{3}\overrightarrow{BC}$

Since they contain a common point ($B$), $A$, $B$, $C$ are collinear.

**2** The sides of the triangle are given by the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$

$\overrightarrow{AB} = (2\mathbf{i} + 2\mathbf{j}) - (5\mathbf{i} - \mathbf{j} + 6\mathbf{k})$

$= -3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$\overrightarrow{AC} = (-3\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}) - (5\mathbf{i} - \mathbf{j} + 6\mathbf{k})$

$= -8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Now $\overrightarrow{AB} \cdot \overrightarrow{AC} = (-3\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \cdot (-8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$

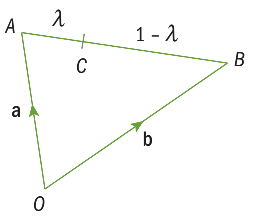
$= (-3 \times -8) + (3 \times -4) + (-6 \times 2)$

$= -24 - 12 - 12$

$= 0$

Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$, the vectors are perpendicular. Hence, $A$, $B$ and $C$ form a right-angled triangle.

**3**




$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{AC} = \overrightarrow{OB} + (1 - \lambda)\overrightarrow{BA}$

$= \mathbf{b} + (1 - \lambda)(\mathbf{a} - \mathbf{b})$

$= \lambda\mathbf{b} + (1 - \lambda)\,\mathbf{a}$

**4** $\mathbf{a} + \mathbf{b} = \begin{pmatrix}5+1\\-1+3\\-3+(-5)\end{pmatrix} = \begin{pmatrix}6\\2\\-8\end{pmatrix}$

$\mathbf{a} - \mathbf{b} = \begin{pmatrix}5-1\\-1-3\\-3-(-5)\end{pmatrix} = \begin{pmatrix}4\\-4\\2\end{pmatrix}$

$(\mathbf{a} - \mathbf{b}).(\mathbf{a} + \mathbf{b}) = \begin{pmatrix}6\\2\\-8\end{pmatrix}.\begin{pmatrix}4\\-4\\2\end{pmatrix} = 24 - 8 - 16 = 0$

$\Rightarrow \mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ are perpendicular.

**5** We need $s$ & $t$ so that

$7s = 3 + 2t$ (1)

$6 + 3s = 1 + 4t$ (2)

$-1 + s = 2 - t$ (3)

(3) $\Rightarrow s = 3 - t$

(1) $\Rightarrow 7(3 - t) = 3 + 2t$

$21 - 7t = 3 + 2t$

$18 = 96$

$t = 2$

(3) $\Rightarrow s = 3 - 2 = 1$

check in (2) $\Rightarrow 6 + 3(1) = 1 + 4(2)$

$9 = 9$ so $s$ and $t$ exist.

so $P$ has position vector $\begin{pmatrix}0\\6\\-1\end{pmatrix} + \begin{pmatrix}7\\3\\1\end{pmatrix} = \begin{pmatrix}7\\9\\0\end{pmatrix}$

Point (7, 9, 0)

**6** **a** $\overrightarrow{AB} = \begin{pmatrix}1\\7\end{pmatrix} - \begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}-3\\2\end{pmatrix} - \begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$

**b** $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{pmatrix}3\\3\end{pmatrix}.\begin{pmatrix}-1\\-2\end{pmatrix} = -3 - 6 = -9$

**c** $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}|\,|\overrightarrow{AC}| \cos B\hat{A}C$

$-9 = \sqrt{3^2 + 3^2}\sqrt{(-1)^2 + (-2)^2} \cos B\hat{A}C$

$-9 = \sqrt{18}\sqrt{5} \cos B\hat{A}C$

$-9 = 3\sqrt{2}\sqrt{5} \cos B\hat{A}C$

$\cos B\hat{A}C = \frac{-3}{\sqrt{2}\sqrt{5}}$

**7** **a** $P$ has position vector

$$\begin{pmatrix}6\\2\\-3\end{pmatrix}+4\begin{pmatrix}-2\\2\\1\end{pmatrix}=\begin{pmatrix}6-8\\2+8\\-3+4\end{pmatrix}=\begin{pmatrix}-2\\10\\1\end{pmatrix}$$

$P$ is $(-2, 10, 1)$

**b** Suppose $\begin{pmatrix}-2\\10\\1\end{pmatrix}=\begin{pmatrix}0\\-12\\7\end{pmatrix}+t\begin{pmatrix}-1\\11\\-3\end{pmatrix}$ for some $t$

$-2=-t$ (1)

$10=-12+11t$ (2)

$1=7-3t$ (3)

$(1)\Rightarrow t=2$

$(2)\Rightarrow 10=-12+11(2)=10$

$(3)\Rightarrow 1=7-3(2)=7-6=1$

so equations are consistent.

so $t = 2$ gives $\begin{pmatrix}-2\\10\\1\end{pmatrix}$ $\therefore P$ lies on $L_2$

**8** **a** $L_2$: $\mathbf{r}=\begin{pmatrix}2\\2\\4\end{pmatrix}+s\begin{pmatrix}1\\7\\3\end{pmatrix}$

**b** $0=\begin{pmatrix}1\\7\\3\end{pmatrix}.\begin{pmatrix}4\\x\\1\end{pmatrix}=4+7x+3=7+7x\Rightarrow x=-1$

**c** $\begin{pmatrix}2\\-3\\-3\end{pmatrix}+t\begin{pmatrix}1\\7\\3\end{pmatrix}=\begin{pmatrix}7\\5\\1\end{pmatrix}+q\begin{pmatrix}4\\-1\\1\end{pmatrix}$

$2+t=7+4q$ (1)

$-3+7t=5-q$ (2)

$-3+3t=1+q$ (3)

**9** Suppose $\mathbf{r}_1=\mathbf{r}_2$

Then $\begin{pmatrix}-4\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}+\mu\begin{pmatrix}-12\\5\end{pmatrix}$

$-4+4\lambda=4-12\mu$ (1)

$3+17\lambda=9+5\mu$ (2)

$(1)\Rightarrow 4\lambda=8-12\mu$

$\lambda=2-3\mu$

$(2)\Rightarrow 17(2-3\mu)=6+5\mu$

$34-51\mu=6+5\mu$

$28=56\mu$

$\mu=\frac{1}{2}$

$(1)\Rightarrow \lambda=2-\frac{3}{2}=\frac{1}{2}$

So ships collide after $\frac{1}{2}$ hour, ie 12.30pm.

collide at $\begin{pmatrix}-4\\3\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix}$

**b** At 12.15, A has position $\begin{pmatrix}-4\\3\end{pmatrix}+\frac{1}{4}\begin{pmatrix}4\\17\end{pmatrix}=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}$

so after 12:15, A's position given by

$\mathbf{r}_1=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}+t\begin{pmatrix}16\\17\end{pmatrix}$ where $t$ is time after 12:15

At 12.30, A's position is

$\mathbf{r}_1=\begin{pmatrix}-3\\\frac{29}{4}\end{pmatrix}+\frac{1}{4}\begin{pmatrix}16\\17\end{pmatrix}=\begin{pmatrix}1\\\frac{23}{2}\end{pmatrix}$

Distance is $\begin{pmatrix}-2\\\frac{23}{2}\end{pmatrix}-\begin{pmatrix}1\\\frac{23}{2}\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}$

so ships are 3km apart.