The Lucas Numbers

Introduction

I did my exploration on Lucas numbers because different series fascinate me and it was related to the Fibonacci numbers which is pretty well known to all the mathematicians across the world so I wanted to find out about the Lucas numbers which are similar to Fibonacci numbers. I also like exploring series, looking and analyzing the patterns. I was also interested in the golden ratio because I have heard it many times and did not really know what it was.

On this exploration I will explore some links between the Lucas numbers and the Fibonacci numbers. I will also obtain the Binet's formula for the Fibonacci numbers using the golden ratio and use this to prove a conjecture made connecting the Fibonacci numbers and the Lucas numbers. Towards the end I will also show some interesting patterns with the Lucas numbers and the Fibonacci numbers and the Pascal's triangle.

Exploration

The French mathematician, Edward Lucas (1842-1891), named the series 0,1,1,2,3,5,8,13... the Fibonacci numbers, came up with another series 2,1,3,4,7,11,18...; which follows the Fibonacci recursive formula but has different initial values. The series is called, Lucas numbers. It is as follows:

$$L_n = L_{n-1} + L_{n-2}$$

$$L_0 = 2$$

$$L_1 = 1$$

Here are some values of the series in comparison to Fibonacci number:

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**...**
L_n	2	1	3	4	7	11	18	29	...
F_n	0	1	1	2	3	5	8	13	...

The Lucas numbers have many similar properties to Fibonacci numbers. They frequently occur in various formulae for Fibonacci number.

In Fibonacci numbers, if p is a factor of q then the Fibonacci number F_p and F_q are also factors. For example, 3 is a factor of 6, therefore $F_{(3)}=2$ is also a factor of $F_{(6)}=8$.

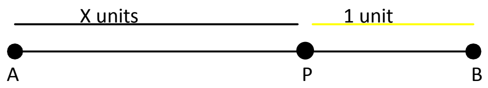
Now we will look at the Fibonacci number at even arrangement, which is $F_{(2n)}$, they should all be divisible by $F_{(2)}$. $F_{(2)}$ is one which a factor of all the integers, so it not attention-grabbing. So we will investigate their $F_{(n)}$ factors (since n is factor of 2n).

n	$F_{(n)}$	2n	$F_{(2n)}$	$k = \frac{F_{(2n)}}{F_{(n)}}$
1	1	2	1	1
2	1	4	3	3
3	2	6	8	4
4	3	8	21	7
5	5	10	55	11
6	8	12	144	18
7	13	14	377	29

We can without difficulty spot that value of k is the Lucas Numbers. And we can make a conjecture relating the Fibonacci numbers and the Lucas numbers.

$$F_{(2n)} = F_{(n)} \times L_{(n)}$$

To prove that conjecture first we have to find a formula to find out F $_{(n)}$ without using other Fibonacci numbers. For that we need the Golden ration (ϕ) which is defined $\frac{AB}{AP} = \frac{AP}{AB}$ for the given line segment below.



$$\frac{x+1}{x} = \frac{x}{1}$$

$$x + 1 = x^2$$

$$x^2 - x - 1 = 0$$

Using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$\phi = \frac{1 \pm \sqrt{5}}{2}$; We can discard the negative value as the length of the side cannot be negative.

$$\phi^2 = \left\{\frac{1+\sqrt{5}}{2}\right\}^2$$
$$= \frac{1 + 2\sqrt{5} + 5}{4}$$
$$= \frac{6 + 2\sqrt{5}}{4} = \frac{3 + \sqrt{5}}{2}$$
$$But\ \phi + 1 = \frac{1+\sqrt{5}}{2} + \frac{2}{2}$$
$$= \frac{3+\sqrt{5}}{2}$$

If we relate that to Fibonacci numbers then we can say that $\phi^2 = F_{(2)}\phi + F_{(1)}$.

Let us try some other values:

$$\phi^3 = \left\{\frac{1+\sqrt{5}}{2}\right\}^3$$
$$= \left\{\frac{1^3 + 3 \times 1^2 \times \sqrt{5} + 3 \times 1 \times \sqrt{5}^2 + \sqrt{5}^3}{8}\right\}$$
$$= \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5}}{8}$$

$$
\begin{aligned}
&= \frac{16 + 8\sqrt{5}}{8} \\
&= 2 + \sqrt{5} \\
&= 2 \times \frac{1 + \sqrt{5}}{2} + 1 \\
2\phi + 1 &= \frac{2 + 2\sqrt{5}}{2} + 1 \\
&= \left(1 + \sqrt{5}\right) + 1 \\
&= 2 + \sqrt{5}
\end{aligned}
$$

$$\phi^3 = F_3\phi + F_2$$

Now we can make a conjecture that:

$$\phi^n = F_n\phi + F_{n-1}$$

Now we are going to prove this conjecture using induction.

$$P_n: \phi^n = F_n\phi + F_{n-1}$$

Working above shows that P_2 and P_3 are correct.

Assume P_n is true for n=k; $P_k: \phi^k = F_k\phi + F_{k-1}$

When n=k+1

$$
\begin{aligned}
&LHS = \phi^{k+1} \\
&= (\phi^k)\phi \\
&= (F_n\phi + F_{k-1})\phi \\
&= F_n\phi^2 + (F_{k+1} - F_k)\phi \\
&= F_k(\phi^2 - \phi) + F_{k+1}\phi \\
&= F_k(\phi + 1 - \phi) + F_{k+1}\phi \\
&= F_{k+1}\phi + F_k \\
&RHS = F_{(k+1)}\phi + F_{k+1-1} \\
&= F_{k+1}\phi + F_k = LHS \\
&\phi^{\mathrm{n}} = \mathrm{F_n}\phi + \mathrm{F_{n-1}}
\end{aligned}
$$

$$F_{k+1} = F_k + F_{k-1}$$
$$F_{k-1} = F_{k+1} - F_k$$

$$\phi^2 = \phi + 1$$

thereby showing that indeed *P*(*k* + 1) holds.
Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that *P*(*n*) holds for all natural n. Q.E.D.

Now let us look at a different value of ratio. We will look at negative inverse of golden ratio

$$
\begin{aligned}
\frac{-1}{\phi} &= \frac{-1}{\left\{\dfrac{1 + \sqrt{5}}{2}\right\}} \\
&= \frac{-2}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}} \\
&= \frac{-2 + 2\sqrt{5}}{\left(1 - \sqrt{5}\right)\left(1 + \sqrt{5}\right)}
\end{aligned}
$$

$$= \frac{-2\left(1-\sqrt{5}\right)}{1-\sqrt{5}^{2}}$$
$$= \frac{-2\left(1-\sqrt{5}\right)}{-4}$$
$$= \frac{1-\sqrt{5}}{2}$$

If we look carefully we have the same value as - ϕ. That is interesting because we discarded the value because the value of the length of that line segment cannot be negative. Now let's see if there is some interesting conjecture like we had with ϕ.

$$F_2\left\{\frac{-1}{\phi}\right\} + F_1 = 1 \times \left\{\frac{1-\sqrt{5}}{2}\right\} + 1$$
$$= \frac{1-\sqrt{5}}{2} + \frac{2}{2}$$
$$= \frac{3-\sqrt{5}}{2}$$
$$\left\{\frac{-1}{\phi}\right\}^2 = \left\{\frac{1-\sqrt{5}}{2}\right\}^2$$
$$= \frac{1-2\sqrt{5}+5}{4}$$
$$= \frac{6-2\sqrt{5}}{4}$$
$$= \frac{3-\sqrt{5}}{2} = F_2\left\{\frac{-1}{\phi}\right\} + F_1$$

We seem to be having the same pattern; let's try some more values of n=3.

$$\left\{\frac{-1}{\phi}\right\}^3 = F_3\left\{\frac{-1}{\phi}\right\} + F_2$$
$$RHS = \left\{\frac{1-\sqrt{5}}{2}\right\}^3$$
$$= \frac{1-3\sqrt{5}+3\times 5-5\sqrt{5}}{8}$$
$$= \frac{16-8\sqrt{5}}{8}$$
$$= 2-\sqrt{5}$$
$$LHS = 2\times\frac{1-\sqrt{5}}{2} + 1$$
$$= 1-\sqrt{5}+1$$
$$= 2-\sqrt{5} = RHS$$

Conjecture $\left\{\frac{-1}{\phi}\right\}^n = F_n\left\{\frac{-1}{\phi}\right\} + F_{n-1}$ also seems to be true for $n \in Z^+$

Now we will prove that conjecture with induction.

$$P_n\colon \phi^n = F_n\phi + F_{n-1}$$

Working above shows that P_2 and P_3 are correct.

Assume P_n is true for n=k; $P_k\colon \left\{\frac{-1}{\phi}\right\}^k = F_k\left\{\frac{-1}{\phi}\right\} + F_{k-1}$

When n=k+1

$$
\begin{aligned}
LHS &= \left\{\frac{-1}{\phi}\right\}^{k+1} \\
&= \left(\left\{\frac{-1}{\phi}\right\}^k\right)\left\{\frac{-1}{\phi}\right\} \\
&= \left(F_n\left\{\frac{-1}{\phi}\right\} + F_{k-1}\right)\left\{\frac{-1}{\phi}\right\} \\
&= F_n\left\{\frac{-1}{\phi}\right\}^2 + (F_{k+1} - F_k)\left\{\frac{-1}{\phi}\right\} \\
&= F_k\left(\frac{-1^2}{\phi} - \frac{-1}{\phi}\right) + F_{k+1}\phi \\
&= F_k\left(\left\{\frac{-1}{\phi}\right\} + 1 - \left\{\frac{-1}{\phi}\right\}\right) + F_{k+1}\left\{\frac{-1}{\phi}\right\} \\
&= F_{k+1}\left\{\frac{-1}{\phi}\right\} + F_k \\
RHS &= F_{(k+1)}\left\{\frac{-1}{\phi}\right\} + F_{k+1-1} \\
&= F_{k+1}\left\{\frac{-1}{\phi}\right\} + F_k = LHS
\end{aligned}
$$

$$F_{k+1} = F_k + F_{k-1}$$
$$F_{k-1} = F_{k+1} - F_k$$

$$\left\{\frac{-1}{\phi}\right\}^2 = \frac{-1}{\phi} + 1$$

thereby showing that indeed $P(k+1)$ holds.
Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $P(n)$ holds for all natural n. Q.E.D.

$\phi^n = F_n\phi + F_{n-1}$ *- Equation 1(Page 3)*
$\left\{\frac{-1}{\phi}\right\}^n = F_n\left\{\frac{-1}{\phi}\right\} + F_{n-1}$ *-Equation 2*
We subtract *equation 2* from *equation 1.*

$$
\begin{array}{rcl}
\phi^n & = & F_n\phi + F_{n-1} \\
-\left\{\frac{-1}{\phi}\right\}^n & = & -F_n\left\{\frac{-1}{\phi}\right\} - F_{n-1} \\
\hline
\end{array}
$$

$$
\begin{aligned}
\phi^n + \left\{\frac{1}{\phi}\right\}^n &= F_n\phi + F_n\left\{\frac{1}{\phi}\right\} \\
\phi^n + \left\{\frac{1}{\phi}\right\}^n &= F_n\left\{\phi + \frac{1}{\phi}\right\} \\
\frac{\phi^{2n} + 1}{\phi^n} &= F_n\left\{\frac{\phi^2 + 1}{\phi}\right\} \\
F_n &= \left\{\frac{\phi^{2n} + 1}{\phi^n}\right\}\left\{\frac{\phi}{\phi^2 + 1}\right\} \\
F_n &= \left\{\frac{\phi^{2n}}{\phi^n} + \frac{1}{\phi^n}\right\}\left[\frac{\frac{1+\sqrt{5}}{2}}{\left\{\frac{3+\sqrt{5}}{2}\right\} + 1}\right]
\end{aligned}
$$

$$F_n = (\phi^n + \phi^{-n})\left[\frac{1+\sqrt{5}}{2} \times \frac{2}{5+\sqrt{5}}\right]$$
$$F_n = (\phi^n - (-\phi)^{-n})\left\{\frac{1+\sqrt{5}}{5+\sqrt{5}} \times \frac{5-\sqrt{5}}{5-\sqrt{5}}\right\}$$
$$F_n = \left(\phi^n - \left\{\frac{-1}{\phi}\right\}^{-n}\right)\left\{\frac{5-\sqrt{5}+5\sqrt{5}-5}{5^2-\sqrt{5}^2}\right\}$$
$$F_n = \left(\left\{\frac{1+\sqrt{5}}{2}\right\}^n - \left\{\frac{1-\sqrt{5}}{2}\right\}^{-n}\right)\left\{\frac{4\sqrt{5}}{20}\right\}$$
$$F_n = \left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^{-n}}{2^n}\right)\left\{\frac{1}{\sqrt{5}}\right\}$$

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^{-n}}{2^n\sqrt{5}}$$
$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$$

This relationship which we just derived of Fibonacci numbers with golden ratio is called Binet's formula. It can be used to find value of a Fibonacci number without having to know the value of last two consecutive numbers.

To find a formula of the Lucas numbers with the Golden ratio; we cannot derive it because it is a closed formula. So we are going to look at the patterns with Golden ratio.

n	ϕ^n	$\left\{-\frac{1}{\phi}\right\}^n$	$\phi^n + \left\{-\frac{1}{\phi}\right\}^n$
0	1.00000	1.00000	2.00000
1	1.61803	-0.61803	1.00000
2	2.61803	0.38197	3.00000
3	4.23607	-0.23607	4.00000
4	6.85410	0.14590	7.00000
5	11.09017	-0.09017	11.00000
6	17.94427	0.05573	18.00000

We can see that the sum is equal to the Lucas numbers. Hence we have a formula of the Lucas numbers.

$$L_n = \phi^n + \left\{-\frac{1}{\phi}\right\}^n$$

$$L_n = \phi^n + \{-\phi\}^{-n}$$

Now we will prove $P_k: F_{(2k)} = F_{(k)} \times L_{(k)}$ using mathematical induction.

for n=1; $F_{2\times1}=F_1\times L_1$
1=1×1
1=1
assume that $P_k: F_{(2k)} = F_{(k)} \times L_{(k)}$ for $n \in Z^+$; then $P_k: F_{2(k+1)} = F_{(k+1)} \times L_{(k+1)}$should be true.

$$RHS = \left(\frac{\phi^{n+1} - (-\phi)^{-n-1}}{\sqrt{5}}\right) \times \left(\phi^{n+1} + \{-\phi\}^{-n-1}\right)$$
$$= \frac{1}{\sqrt{5}}\left\{\phi^{(n+1)}\phi^{n+1} + (-\phi)^{(-n-1)}\,\phi^{n+1} - (-\phi)^{-n-1}\phi^{n+1} - (-\phi)^{-n-1}(-\phi)^{-n-1}\right\}$$
$$= \frac{1}{\sqrt{5}}\left\{\phi^{2(n+1)} - (-\phi)^{2(-n-1)}\right\}$$
$$= \frac{\left\{\phi^{2(n+1)} - (-\phi)^{2(-n-1)}\right\}}{\sqrt{5}}$$
$$LHS = \frac{\phi^{2(n+1)} - (-\phi)^{-2(n+1)}}{\sqrt{5}}$$
$$= \frac{\phi^{2(n+1)} - (-\phi)^{2(-n-1)}}{\sqrt{5}} = RHS$$

thereby showing that indeed $P(k + 1)$ holds.
Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that $P(n)$ holds for all natural n. Q.E.D.

I found some other interesting patterns that are worth investigating are listed below.

If we look at the table we can see that sum of alternate Fibonacci numbers is Lucas number:

n	0	1	2	3	4	5	6	7	...
L_n	2	1	3	4	7	11	18	29	...
F_n	0	1	1	2	3	5	8	13	...

This provides us with the first equation connecting Fibonacci numbers (F_n) and Lucas numbers (L_n).

$$L_n = F_{n-1} + F_{n+1}; n \in R$$

However if we add, subtract, divide or multiply; alternate Lucas number it does not give us Fibonacci numbers. The sum of L_2 and L_4 is not F_3.

Nevertheless, we can see another patter if we add up a few more numbers.

n	L_{n-1}	L_{n+1}	$L_{n-1} + L_{n+1}$	F_n
1	2	3	5	1
2	1	4	5	1
3	3	7	10	2
4	4	11	15	3
5	7	18	25	5

As we can see the pattern, we have found another equation connecting Lucas number (L_n) and Fibonacci numbers (F_n).

$$F_n = \frac{L_{n-1} + L_{n+1}}{5}$$

$$5F_n = L_{n-1} + L_{n+1}$$

We know that Fibonacci numbers appear in Pascal's triangle as the sum of the diagonals. I found a method to get value of the Lucas numbers from Pascal's triangle. Here is an alternative version of Pascal's triangle

n	0	1	2	3	4	5	6	7	8
0	1	-	-	-	-	-	-	-	-
1	-	1	1	-	-	-	-	-	-
2	-	-	1	2	1	-	-	-	-
3	-	-	-	1	3	3	1	-	-
4	-	-	-	-	1	4	6	4	1
5	-	-	-	-	-	1	5	10	5
6	-	-	-	-	-	-	1	6	15
7	-	-	-	-	-	-	-	1	7
8	-	-	-	-	-	-	-	-	1
F_n	1	1	2	3	5	8	13	21	34

As we can see that the sum of columns is Fibonacci numbers. For Lucas numbers also we have to add the columns and the rows but for each number in the column we have to multiply it by its column number and divide by its row number. For example, let's take column 3; we multiply 2 by 3 and divide by 2 and we multiply 1 by 3 and divide by 3 and we add the result. We should get L_3 which is 4.

$$
\begin{aligned}
&= 2 \times \frac{3}{2} + 1 \times \frac{3}{3} \\
&= 3 + 1 \\
&= 4 = L_3
\end{aligned}
$$

Here is a screen shot of a spreadsheet of excel getting the Lucas numbers from the method above.

	0	1	2	3	4	5	6	7	8	9
0	1									
1		1x1/1=1	1x2/1=2							
2			1x2/2=1	2x3/2=3	1x4/2=2					
3				1x3/3=1	3x4/3=4	3x5/3=5	1x6/3=2			
4					1x4/4=1	4x5/4=5	6x6/4=9	4x7/4= 7	1x8/4= 2	
5						1x5/5=1	5x6/5=6	10x7/5=14	10x8/5=16	...
6							1x6/6=1	6x7/6= 7	15x8/6=20	...
7								1x7/7= 1	7x8/7= 8	...
8									1x8/8= 1	...
		1	3	4	7	11	18	29	47	...

Pic 1 Obtaining Lucas numbers using Pascal's Traingle[1]

When I was doing my research about Lucas numbers and it relationship with Fibonacci numbers I found out about other things also like its relationship with Pascal's triangle and use of Golden ratio. I was interested in investigating them further but I cannot because I do not have enough room and time.

I did enjoy doing all the research for Lucas numbers and Fibonacci numbers. In particular, I found it very stimulating to work on the same concepts as these mathematicians did without any use of technology or guidance. I respect them for their creative ways of thinking. I found fulfilling that I was able to prove the conjecture and derive the Binet's formula on my own and I got appropriate results.

This experience gave me a chance to use my mathematics skills and put them in a practical situation. In addition to that I got to know more about things, like Golden ratio, which I was not familiar with before.

Sources

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