The Legend (Introduction)

The Tower of Hanoi (also known as The Tower of Brahma and The End of The World Puzzle) was invented in 1883 by Edouard Lucas, a French mathematician. It is said that he designed the puzzle based on a legend of a Hindu Temple. In the beginning of time the priests in the temple were given a tower of 64 gold disks, each smaller in size then the disk beneath. They were to transfer the disks from one of three poles to another without allowing any disk to be placed on top of a smaller one (as the weight from the disk will crush the one beneath it). It is said that when the day the priests successfully transfer the 64 disks from one pole to another, the world will crumble and vanish. If this legend was true, could there be a way of predicting the end of the world?

A Good introduction

The Tower of Hanoi is a classic puzzle for all ages as the number of disks creates endless levels of difficulty and fun. Though the aim of this game is simple, it reveals many mathematical concepts and patterns through the process of playing the puzzle. These patterns will be explored and analyzed and the legend of The Tower of Hanoi will be put to the test.

A Aim identified

Finding a Pattern

The task is to find out how many moves it takes 64 disks to transfer from one pole to another pole of the Tower of Hanoi. Solving the puzzle using a smaller number of disks will be easier to analyze and understand. Let's look at how the Tower of Hanoi is solved using 1, 2, and 3 discs.

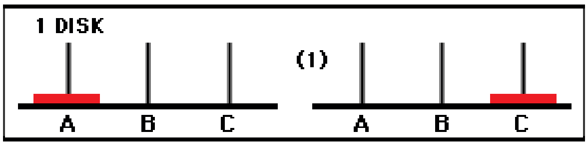


Figure A. One Move

Moves 1: move disk 1 to post c

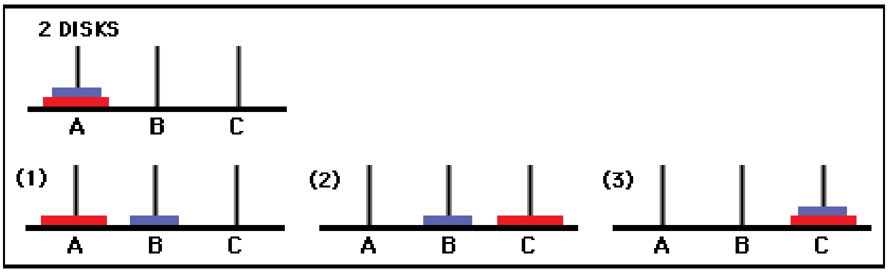


Figure B. Three Moves

Move 1: move disk 2 to post B

Move 2: move disk 1 to post C

Move 3: move disk 2 to post C

B *These diagrams are helpful, but better clarity could be achieved by identifying which is disk 1 and 2.*

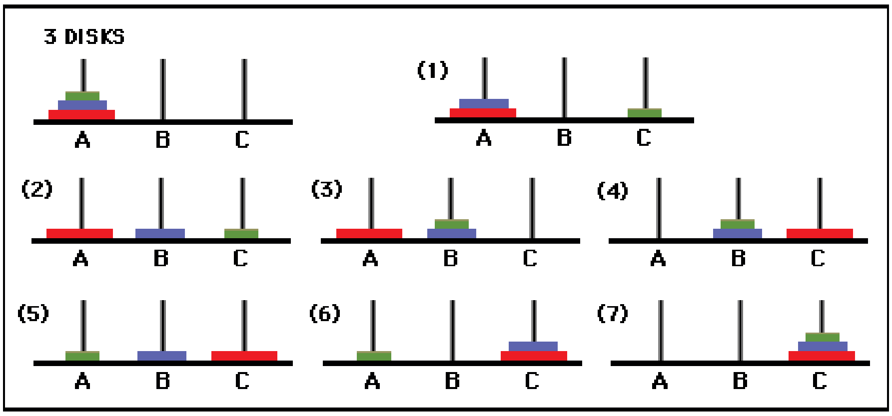


Figure C. Seven Moves

Move 1: move disk 3 to post C

Move 2: move disk 2 to post B

Move 3: move disk 3 to post B

Move 4: move disk 1 to post C

Move 5: move disk 3 to post A

Move 6: move disk 2 to post C

Move 7: move disk 3 to post C

The number of moves it takes to complete the puzzle for an amount of discs is a *recursive pattern*. A recursive pattern requires information from the previous term of the pattern to determine the following term. In this situation, it is the number of moves required for n discs to transfer from A to C.

A *Unsure what this refers to*

Figure 1-3 shows that one must first transfer n-1 discs (n being the number of discs in the puzzle) to pole B. In Figure C it takes three moves to transfer two discs to pole B (step 3 of Figure C). The number of moves required shall be the variable M.

The next step is to transfer the remaining disc from A to C (step 4 of Figure C).

The last step is to transfer the discs from pole B to C. It will require the same number of moves required in the first step also known as the variable M. In Figure C it takes three moves to transfer n-1 discs from pole B to pole C (step 7 of Figure C).

Reflecting on the steps you can see that to solve the puzzle you move M amount of moves twice (from A to B, and from B to C). You also make a single move from A to C. Mathematically the recursive pattern should look like this:

# of discs	Total Moves	Equation 2M + 1
1	1	2(0) + 1 = **1**
2	3	2(**1**) + 1 = **3**
3	7	2(**3**) + 1 = 7

Table A.

B *Not an equation*

In Table A under the Equation column, you are able to clearly see why there is a recursive pattern present. The total moves from the 1 disc puzzle become the M of the second puzzle as shown in the bold and black numbers. The total moves for the 2 disc puzzle become the M of the third puzzle as shown in the bold and grey numbers. Using the recursive pattern, you are able to find the number of moves it takes for a 4, 5 disc puzzle and for an n disc puzzle as shown in Table B on page 4.

C *More diagrams would help here.*

# of discs	Total Moves	Equation 2M + 1

1	1	2(0) + 1 = **1**
2	3	2(**1**) + 1 = **3**
3	7	2(**3**) + 1 = **7**
4	15	2(**7**) + 1 = **15**
5	31	2(**15**) + 1 = 31
n	T	2(M) + 1 = T

Table B.

E Poor display of knowledge—not a recognized geometric sequence in differences

Referring back to the original task, the number of moves it takes to solve a 64 disc puzzle is still unknown. It is possible to use the recursive pattern and equation 2M + 1 to find the answer but in order to do this one needs to know what M is. This reveals a weakness in recursive patterns because in order to know what M is for 64 discs, one must know M for 63 discs, M for 62 discs, and so on. This process would be very time consuming and therefore not an efficient way of solving the problem.

D Some reflection here

Another problem with using the recursive pattern is if you wanted to graph the data to compare the # of discs with the Total Moves. If one wanted to use the recursive equation 2M + 1 one would discover that it is not possible. This is because of the nature of a recursive equation. The M variable is unknown unless the previous terms are known. Therefore it is not possible to graph data using a recursive equation.

Perhaps a non-recursive way of looking at the puzzle would be more efficient and useful. Using the data obtained from the recursive pattern we can find an *explicit pattern* in the puzzle. An explicit pattern is basically a non-recursive pattern meaning it doesn't need information from previous terms to obtain an answer.

E Explicit versus recursive recognizes two important mathematical ideas related to patterns.

Creating a Formula

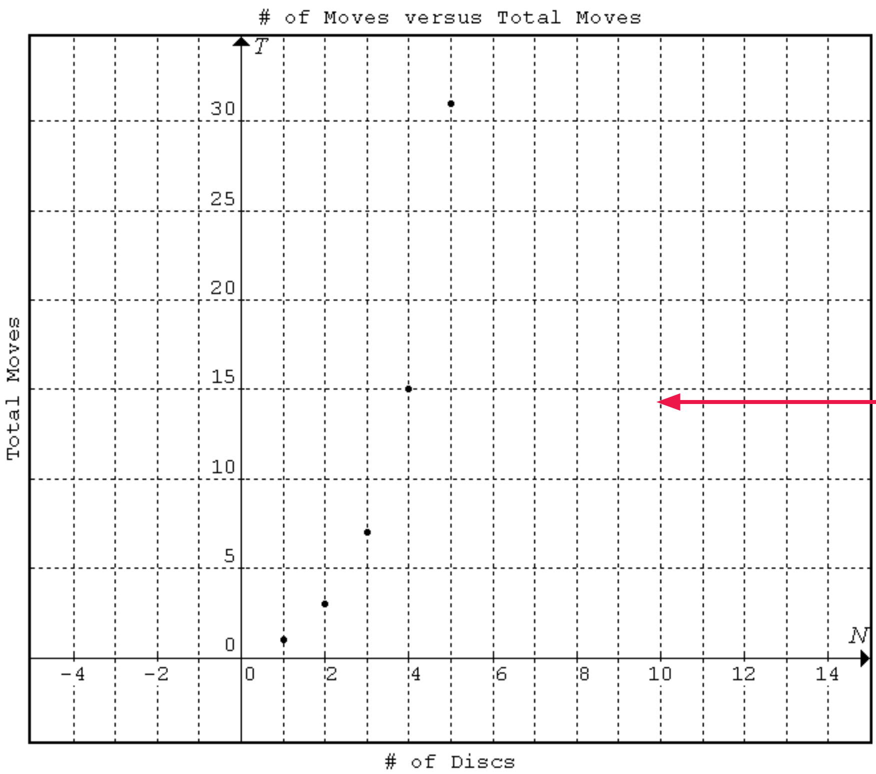
First, take the values from the first two columns of Table B. These values are displayed below.

# of discs	Total Moves
1	1
2	3
3	7
4	15
5	31
n	T

Table C

C Trying to find a formula by using graphical means indicates independent thinking.

At first glance, there doesn't seem to be an explicit relationship between # of discs and total moves. One way to find a relationship between the values is by guess-and-test until a working function is made. But this method can be very time consuming and inefficient like the previous one. A more efficient way to finding a function for two sets of data is by graphing them. Below is a graph of # of discs versus total moves.

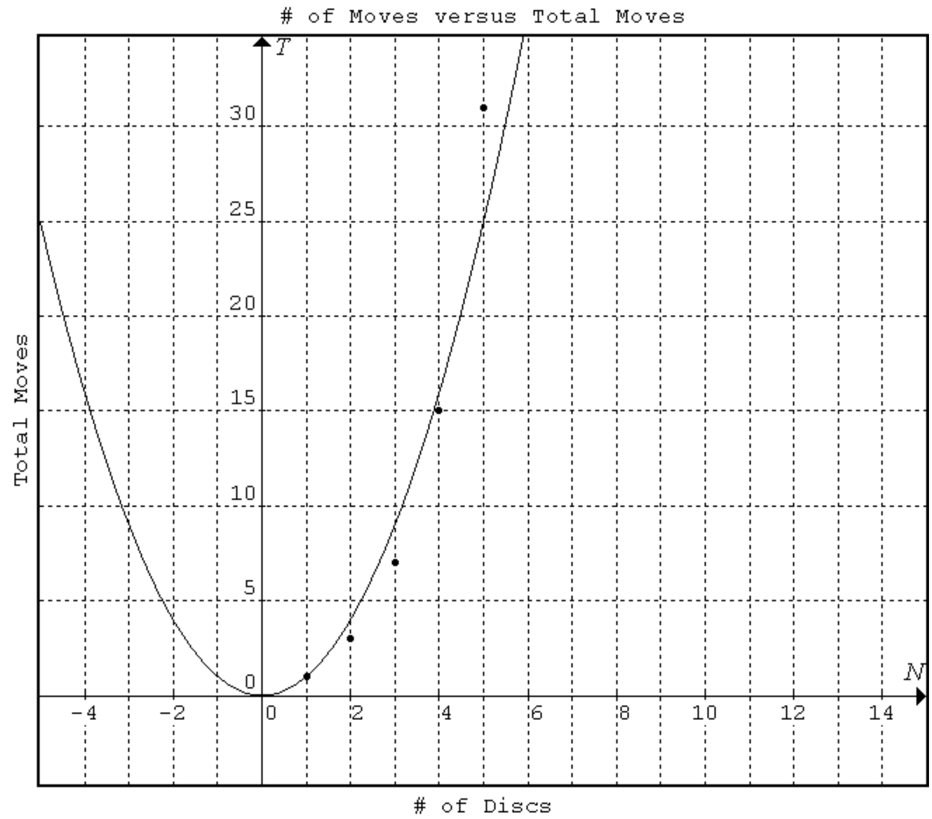


B Well-labelled graph; however, student should try to avoid wasted open space by adjusting the domain.

Graph A.

Graph A appears to have a trend which suggests there is a relationship between the data graphed. From past knowledge of common functions, we can try different functions and compare them to determine which is best suited for this graph. Graph A has a growth trend because the points move towards the right and upwards. There are two common functions that show a similar shape to Graph A. The functions are graphed and compared to the points in Graph A below.

Graph B compares the data points with the quadratic function $y=x^2$.



Graph B.

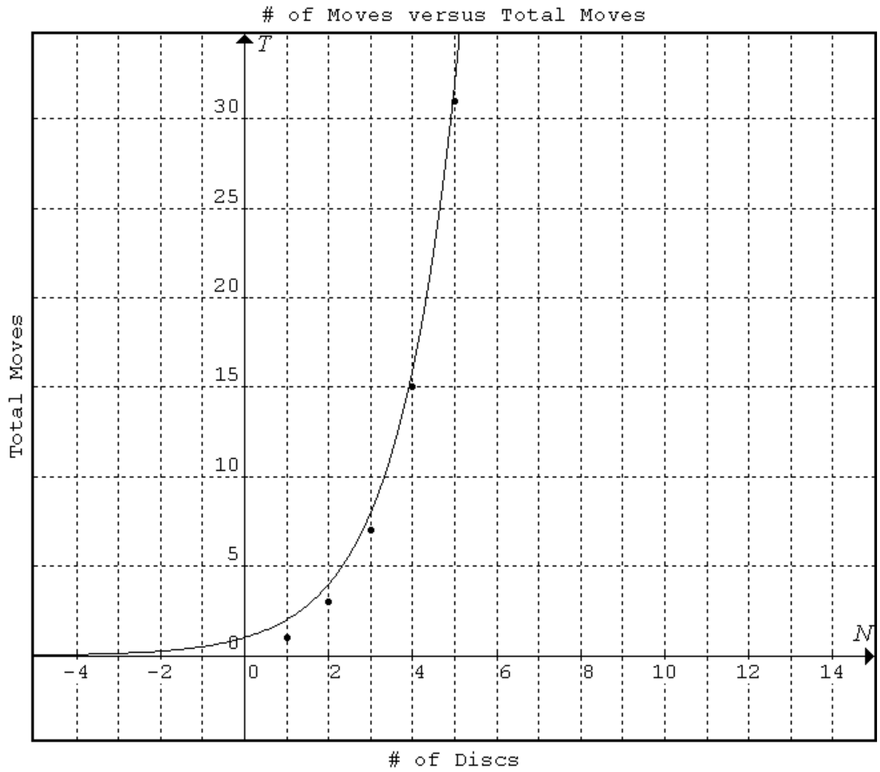
In graph B, the quadratic function $y=x^2$ has a similar trend to the data points except the quadratic function forms a parabola. This is unsuitable to the data points because they do not form a trend in the shape of a parabola. Even when negative numbers are used (though this is unrealistic as it's impossible to have negative # of moves), the trend is

B Student doesn't acknowledge that discrete data are being modelled with a continuous function.

E Confusing why negatives are considered here

E Not good knowledge demonstrated—seems as though $y=x^2$ is only possible parabola considered still not that of a parabola. Therefore the quadratic function is not a function of the data points.

Graph C compares the data points with the exponential function $y=2^x$.



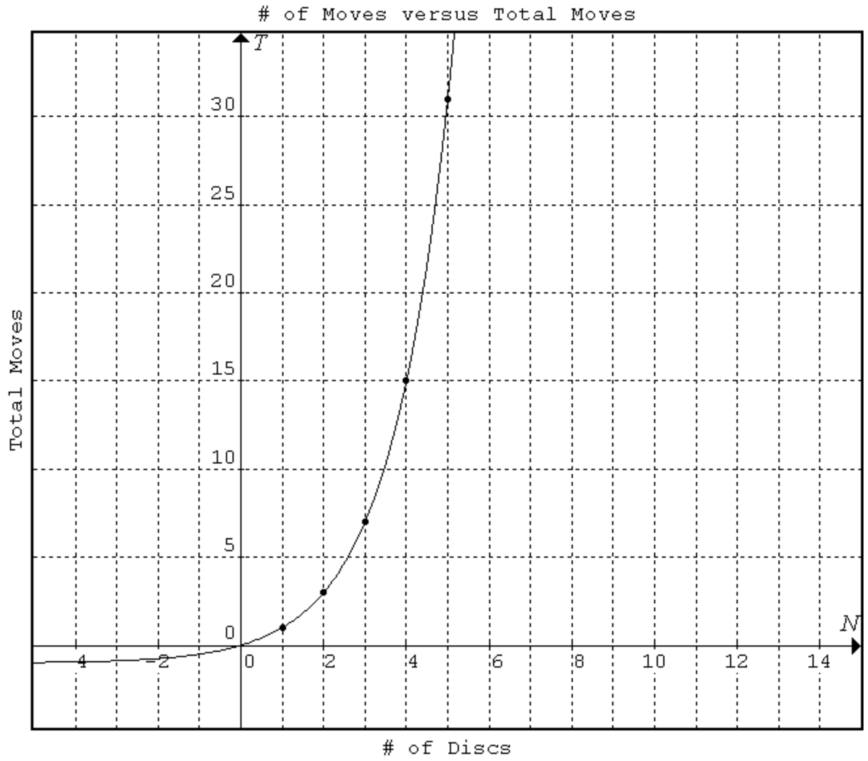
Graph C.

In Graph C, the exponential function $y=2^x$ has a very similar trend to the trend of the data points. It is important to note that the exponential function is actually $y=a^x$ where $a > 0$. Using $y=1^x$ would only produce a horizontal line. Therefore the next value used was $y=2^x$.

In order to make the function fit better with the points we make transformations that slightly change the function. The first step is to lower the line so that the line can become closer to the data points. This is done by a transformation of 1 down.

D Good ideas linking recursive formula and graph through transformations

Graph D shows the function $y=2^x - 1$ compared with the data points.



Graph D.

The function in graph D appears to completely fit with the data points after the transformation. This means that there is a possibility that the relation between # of moves and Total moves is $2^x - 1$. We can confirm this hypothesis by calculating the # of Discs with Total Moves using the formula $2^x – 1$.

E *Good link between the pattern and the function*

# of discs (x)	Total Moves (y)	Equation $2^x – 1= y$
1	**1**	$2^1 - 1 =$ **1**
2	**3**	$2^2 - 1 =$ **3**
3	**7**	$2^3 - 1 =$ **7**
4	**15**	$2^4 - 1 =$ **15**
5	**31**	$2^5 - 1 =$ **31**

Table D.

Applying the Formula

The formula for finding the number of moves it takes an amount of discs to move from pole A to C of the Tower of Hanoi is $y = 2^x - 1$ where x is the # of discs and y is the total amount of moves.

Now the formula has been found, it can be applied to a task. In this case, the task is to find the number of moves it takes to transfer 64 disks from pole A to C of the Tower of Hanoi.

Total moves = 2^{64} -1

=$1.844674407 \times 10^{19}$ total moves to transfer 64 discs from A to C

B *Inconsistent terminology—approximate here (without saying so) but uses something for "approximately equal to" later*

To summarize, the legend of the Tower of Hanoi was that when all the discs have been transferred from A to C, the world will come to an end. Suppose one disc was transferred every second, how many years will it take to complete the puzzle?

To solve this question, we take the total moves calculated and divide them by units of time until we obtain the number of years it takes to complete the puzzle.

Total moves=2^{64} -1 = $1.844674407 \times 10^{19}$

1 move = 1 second $\therefore$ 60 moves = 1 minute of time passed

Converting seconds to years:

Total moves/60 seconds/60 minutes/24 hours/365 days= years passed

$1.844674407 \times 10^{19}$ /60/60/24/365 = $5.849424174 \times 10^{11}$ years passed

=584,942,417,400 years passed $\cong$ 580 billion years

B *Not correct use of "approximately equal to" sign*

If one disc was transferred every second, it would take approximately 580 billion years to transfer 64 discs from pole A to C of The Tower of Hanoi.

Reflection

A Rationale identified at end

I chose this topic because I remember playing this simple but challenging game at Science World. While playing the game I did not naturally think of it as a mathematical concept but naturally I would identify patterns in the game as I solve the puzzle and the difficulty increases. I chose this topic because the history of it fascinates me. Also I knew that the game had several mathematical concepts that were I recently learned. I wanted to see if my mathematical knowledge of transformations, functions, and patterns could be put to use and solve the legend behind The Tower of Hanoi.

When exploring the mathematics of this game, I learned to see things in life in a mathematical point-of-view as well as seeing mathematics in a realistic point-of-view. The process of solving the legend showed the relationships that mathematical concepts have with each other as well as with life. I learned that everything is connected mathematically and even the simplest things can be dissected and analyzed in a mathematical way.

D Rather simple reflections

Conclusion

According to the legend of the Tower of Hanoi, if one disc was transferred every second since the beginning of time, it would take about 580 billion years until the puzzle is solved and the world comes to an end. If this were true, then the world still has many more years to live.

This solution was discovered through the recognition of a recursive pattern in the puzzle. Through this recursive pattern the function $y=2^x-1$was created. Using this formula, the number of moves it takes to solve a 64 disc Tower of Hanoi puzzle was obtained. This function is useful for obtaining the number of moves for any amount of discs in The Tower of Hanoi. This is useful when trying to obtain the minimal number of moves to complete the puzzle as a way to challenge one's intellectual strength.

A Good conclusion—we can see the aim is fulfilled, so work is complete.

Words Cited

Bogomolny, A. "Tower of Hanoi from Interactive Mathematics Miscellany and Puzzles." *Interactive Mathematics Miscellany and Puzzles, Index*. Alexander Bogomolny. Web. 04 Jan. 2010. <http://www.cut-the-knot.org/recurrence/hanoi.shtml>.

Dr. Math. "Math Forum: Ask Dr. Math FAQ: Tower of Hanoi." *The Math Forum @ Drexel University*. Drexel University. Web. 04 Jan. 2010. <http://mathforum.org/dr.math/faq/faq.tower.hanoi.html>.

"LHS: Tower of Hanoi Facts." *Lawrence Hall of Science*. Lawrence Hall of Science. Web. 08 Jan. 2010. <http://lawrencehallofscience.org/java/tower/towerhistory.html>.

Weisstein, Eric W. "Tower of Hanoi -- from Wolfram MathWorld." *Wolfram MathWorld: The Web's Most Extensive Mathematics Resource*. Wolfram Research, Inc., 07 Jan. 2010. Web. 08 Jan. 2010. <http://mathworld.wolfram.com/TowerofHanoi.html>.