MATHEMATICS SL

3-D VECTOR EXAMPLE PROBLEMS

1. Given coordinates of A and B in 3-space, express **AB** as:

   (a) a 3x1 column vector
   (b) in the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

   Example: A = (5,7,-2) and B = (8,3,4) . Then

$$\mathbf{AB} = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 8-5 \\ 3-7 \\ 4-(-2) \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = 3\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$$

2. Given coordinates of 3 of 4 vertices of a parallelogram in 3-space, find coordinates of the 4th vertex using **CD** = **-AB** .

   Example: A = (2,3,1) , B = (6,5,4) , C = (3,1,5) ABCD is parallelogram. Find D

$$\mathbf{AB} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \Rightarrow \quad \mathbf{CD} = -4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{OD} = \mathbf{OC} + \mathbf{CD} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k} + (-4\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = -\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \Rightarrow \quad D = (-1,-1,2)$$

3. Calculate sums, differences and scalar multiples of vectors.

   Example: Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}$. Find $4\mathbf{a} - 3\mathbf{b} + 2\mathbf{c}$.

$$4\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - 3\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4(2)-3(4)+2(5) \\ 4(-1)-3(3)+2(4) \\ 4(5)-3(-2)+2(0) \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 26 \end{pmatrix}$$

4. Find the magnitude (length) (size) of a vector.

   Example: $\mathbf{a} = \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$ Then $\left|\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}\right| = \sqrt{2^2 + 6^2 + (-3)^2} = \sqrt{49} = 7$

5. Find a unit vector (a vector of unit magnitude) in a given direction.

Example: Find a unit vector in the direction of the vector $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

From previous example, magnitude = 7 so unit vector is $\frac{1}{7}\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 2/7 \\ 6/7 \\ -3/7 \end{pmatrix}$

6. Given a direction and a magnitude, find the vector.

Example: A jet is flying at 700 km h^{-1} in the direction of $\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$.

Write down its velocity vector.

$$\mathbf{v} = 700\begin{pmatrix} 2/7 \\ 6/7 \\ -3/7 \end{pmatrix} = \begin{pmatrix} 200 \\ 600 \\ -300 \end{pmatrix}$$

7. Express the vector (parametric) equation of a straight line given

(a) the coordinates of one point on the line and a vector in the direction of the line.

(b) the coordinates of two points on the line.

Example: (a) Find vector equation of line through (2,1,3) in direction of 3**i** +5**j** -2**k** .

Equation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + p\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2+3p \\ 1+5p \\ 3-2p \end{pmatrix}$$

Example: (b) Find vector equation of line through A (5,7,-2) and B (8,3,4)

Vector in direction of line is **AB** = 3**i** - 4**j** + 6**k** so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix} + p\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} + p\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}$$

Note: *Students are not responsible for Cartesian equations of lines in 3-D.*

8. Given the equations of two non-parallel 3-D lines in parametric form, either

(a) find the coordinates of their point of intersection, or

(b) show the two lines do not intersect.

Example: (a) Line 1 has the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -4 \end{pmatrix} + s\begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$

Line 2 has the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

At their point of intersection

$$x = 5 + 2s = -5 + 2t \Rightarrow 2s - 2t = -10$$
$$y = 9 + 6s = 0 - t \Rightarrow 6s + t = -9$$
$$z = -4 - 3s = -4 + 2t \Rightarrow 3s + 2t = 0$$

3 equations in 2 unknowns do not usually have a solution. To check, solve 2 of the 3 equations for s and t and then find if this solution also satisfies the other equation. If it does, there is a point of intersection, if it doesn't, there is not.

In the case of our two lines, solving the equations for x and z gives $s = -2$ and $t = 3$. Checking the equations for y shows this solution satisfies all 3 equations. The point of intersection will be:

$x = 5 + 2(-2) = -5 + 2(3) = 1$, $y = 9 + 6(-2) = 0 - 3 = -3$, $z = -4 - 3(-2) = -4 + 2(3) = 2$

so the point of intersection of these 2 lines is (1,-3, 2)

Example: (b) Line 3 has the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

To see if line 1 and line 3 intersect, we follow the same procedure and obtain the same relation between s and t from x and z, but from y we obtain $9 + 6s = 3 - t \Rightarrow 6s + t = -6$. Since $6(-2) + 3 = -9 \neq -6$, we know that these lines do not intersect. Such lines are called ***skew*** lines.

Note that one must use a different parameter for each of the two lines being tested.

9. Write and interpret displacement-time equations in the form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad \text{where} \quad \boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$ is the velocity vector and t is time.

Example: The equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 500 \\ 200 \\ 10 \end{pmatrix} + t\begin{pmatrix} 300 \\ 400 \\ -2 \end{pmatrix}$ could describe an airplane that at

$t = 0$ is 500 km East, 200 km North and 10 km above the reference point and has an East velocity component of 300 km h^{-1}, a North velocity component of 400 km h^{-1} and is descending towards the surface at 2 km h^{-1}. The time in hours is given by t.

10. Calculate the scalar product of two 3-D vectors.

Example: $\boldsymbol{a} = 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$, $\boldsymbol{b} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$\boldsymbol{a} \cdot \boldsymbol{b} = 2(2) + 6(-1) -3(2) = -8$

11. Apply $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ to problems involving perpendicular 3-D vectors.

Example: $\boldsymbol{a} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$; $\boldsymbol{a} + k\boldsymbol{b}$ is perpendicular to $\boldsymbol{b}$. Find k.

$$\begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 4+2k \\ 5+k \\ -3+k \end{pmatrix} = 4(4+2k)+5(5+k)=(-3)(-3+k)=50+10k=0$$

$$\Rightarrow k = -5 \quad \Rightarrow \quad \boldsymbol{a} + k\boldsymbol{b} = \begin{pmatrix} 4-5(2) \\ 5-5(1) \\ -3-5(1) \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix}$$

12. Use the scalar product to find the angle between:

(a) two 3-D vectors in component form

(b) two intersecting 3-D lines in vector (parametric) form.

Example: (a) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$$|\vec{a}| = \sqrt{2^2 + 6^2 + (-3)^2} = 7 \quad , \quad |\vec{b}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{2(2) + 6(-1) + (-3)(2)}{7(3)} = \frac{-8}{21} \quad \Rightarrow \quad \theta = 112.4°$$

Example: (b) In exercise 8(a), Line 1 and Line 2 were shown to intersect at the point (1,-3, 2). A vector in the direction of Line 1 is vector ***a*** above. A vector in the direction of Line 2 is vector ***b*** above. Therefore, the angle between the two lines at their point of intersection is 112.4°.