CIRCULAR FUNCTIONS AND TRIGONOMETRY

Definitions and Formulae

Before making a start on any trigonometric question, check your calculator is in the right mode.

Radians: Radians are an alternative to degrees when measuring the size of angles. Although it is easier to *think* in degrees, radians are often used with trigonometric functions and *must* be used when differentiating or integrating them.

- The conversion is π radians = 180°. (An angle is assumed to be in radians unless the degrees symbol is given).

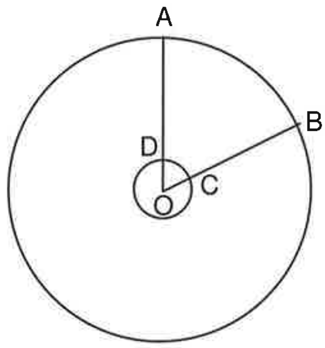
$30° = \pi/6$
$45° = \pi/4$
$60° = \pi/3$
$90° = \pi/2$
$120° = 2\pi/3$
$180° = \pi$
$270° = 3\pi/2$
$360° = 2\pi$

It is worth memorising some key angles in radians (see table on the left). π appears in many angles when expressed in radians (because of the conversion) but it does not have to. For example, 45° = 0.785 rad, but this is not an *exact* conversion, unlike $\pi/4$.

There are two circle formulae which are used when a sector angle is expressed in radians. If the angle is θ and the radius of the circle is r:

- Arc length of sector = $r\theta$
- Area of sector = $\frac{1}{2} r^2\theta$

The diagram shows two concentric circles with radii 1 and 4.
If AOB = $\pi/3$, find
a) The area of ABCD
b) The perimeter of ABCD



AOB is in radians. The area of sector AOB = $\frac{1}{2} r^2\theta = \frac{1}{2} \times 4^2 \times \pi/3$
Similarly, sector DOC has area $\frac{1}{2} \times 1^2 \times \pi/3$
Subtracting, area ABCD = $8\pi/3 - \pi/6$ = 7.85

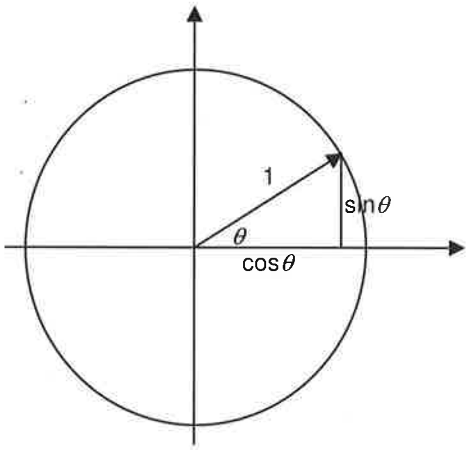
For the perimeter, AD = BC = 3. Arc AB = $r\theta = 4 \times \pi/3$, and arc CD = $1 \times \pi/3$. So, total is: $3 + 3 + 4\pi/3 + \pi/3$ = **11.24**

Learn this table:

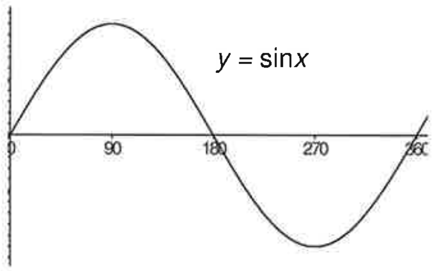
	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

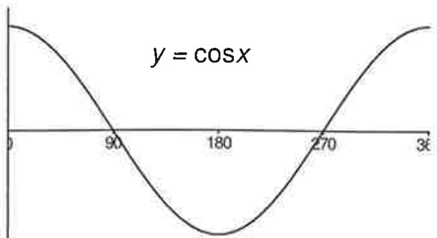
A neat way to remember the exact values for sin (ie the top row of the table) is that they form the series $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

Trigonometric functions:
The diagram shows a circle with radius 1 (a *unit circle*). A line is drawn from the centre to a point on the circumference, and this forms angle θ with the x-axis. Then the x-coordinate of the point is defined as the cosine of the angle ($\cos\theta$) and the y-coordinate as the sine ($\sin\theta$). These definitions will apply as the line rotates full circle, giving the sin and cos for all angles from 0° to 360°. When these are plotted as graphs, we get the following:



These graphs can, of course, be extended to show the sin and cos for *all* angles.





Points to note:

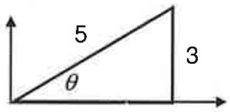
- The range of both functions is $-1 \le f(x) \le 1$
- $\sin x > 0$ for angles between 0° and 180°
- $\cos x > 0$ for angles between 0° and 90°, also between 270° and 360°
- Both functions have a *period* (ie repeat themselves) every 360°.

Simple trigonometric equations: $\sin\theta = 0.4$, $0° \le \theta \le 360°$. What is the value of θ? We want to know what angle has a sin which is 0.4. Using the inverse of the sin function (written as $\sin^{-1}$ or arcsin) on your calculator, $\theta = 23.6°$. Using the symmetry of the sin graph above, another solution is 180 – 23.6 = 156.4°. (If the domain is in radians, you can either work in degrees and convert at the end, or set your calculator to radians: this gives $\theta = 0.412$, and the second solution is $\pi - 0.411 = 2.73$).

Another example: Solve $\cos(\theta - 30) = 0.2$, $0° \le \theta \le 360°$

$\cos^{-1}(0.2) = 78.5°$ or $281.5°$
So $\theta - 30 = 78.5$ or 281.5
$\theta = 108.5°$ or $311.5°$

Finding sin from cos (and cos from sin): A simple trick is to draw a right-angled triangle. eg If $\sin\theta = \frac{3}{5}$, what is $\cos\theta$? Having put 3 as the "opposite" and 5 as the hypoteneuse, the remaining side must be 4. So $\cos\theta = \frac{4}{5}$. If θ was obtuse, $\cos\theta$ would be $-\frac{4}{5}$.



Trigonometric identities: The identities on the right are on your formulae sheets but should be memorised.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

If A is an obtuse angle in a triangle and sinA = $\frac{5}{13}$, calculate the exact value of sin2A.

sin2A = 2sinAcosA, so we need cosA. Using the little trick above, draw a 5, 12, 13 triangle. Hence, cosA = - 12/13.

So, sin2A = 2 × 5/13 × (-12/13) = <u>- 120/169</u> (Note the answer has to be <u>exact</u>).

Solve the equation $3\sin^2 x = \cos^2 x$, for $0° \le x \le 180°$.

First, get everything in terms of $\sin^2 x$, then make $\sin^2 x$ the subject. When you square root, remember the ±. This will effectively give you two equations to solve. But note the domain.

YOU SOLVE

<u>$x = 30°$ or $150°$</u>

Find the exact solutions to the equation sin2x = sinx, for $0 \le x \le 2\pi$

$\sin 2x = \sin x$
$2\sin x\cos x = \sin x$
$2\sin x\cos x - \sin x = 0$ *(as with quadratics, it is important to get 0 on the right hand side)*
$\sin x(2\cos x - 1) = 0$
So $\sin x = 0$ or $2\cos x - 1 = 0 \Rightarrow \cos x = ½$

If $\sin x = 0$, $x = 0°, 180°, 360°$. If $\cos x = ½$, $x = 60°$ or $300°$

So, in radians, <u>$x = 0, \pi/3, \pi, 5\pi/3, 2\pi$</u>

Harder Trigonometric Equations

Equations which lead to quadratics: Consider the following question:

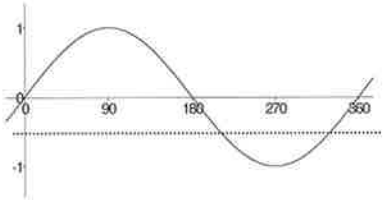
Solve $2\cos^2 x + \sin x = 1$, $0° \le x \le 180°$, *giving answers exactly*

We cannot solve an equation directly with sin *and* cos in it. So, using the identity $\sin^2 x + \cos^2 x = 1$, we get:

$2(1 - \sin^2 x) + \sin x = 1$ which can be rearranged to give:

$2\sin^2 x - \sin x - 1 = 0$

Solving sinx = -0.5



This is a quadratic of equation of the form $2y^2 - y - 1 = 0$. This factorises to give $(2y + 1)(y - 1) = 0$ and thus $y = -0.5$ or 1.

It follows that $\sin x = -0.5$ or 1, giving solutions $x = 210°$, $330°$ or $90°$.

Solving sin$a(x + b) = 0$: If $a = 2$, there will be twice as many solutions in a given range. If $a = 3$, three times as many, and so on. For example, to solve $\sin 2x = 0.73$, $0° \le x \le 360°$

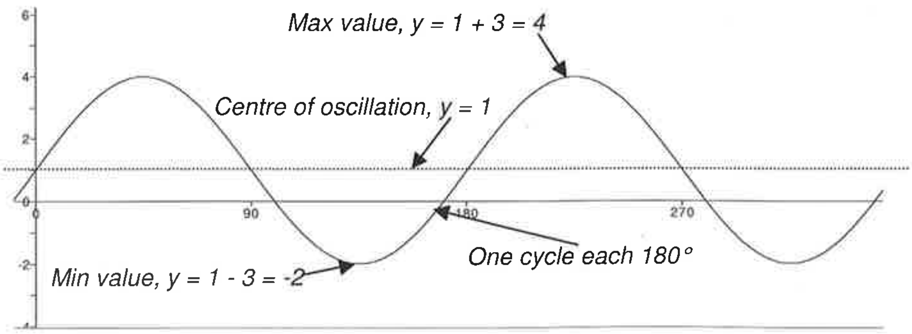
- Calculate which angles have a sin of 0.73… — 46.9°, 133.1°
- Now extend the range of angles by adding 360° …. — 406.9°, 493.1°
- These are values of $2x$. Dividing by 2 gives values of x, and brings the answers into the required range…. — **23.4°, 66.6°, 203.4°. 246.3°**

If the problem had been to solve $\sin 2(x + 20) = 0.73$, the final solutions would be obtained by subtracting 40° from the four given above.

Exactly the same methods are used to solve $\cos a(x + b) = 0$.

Transformations of trigonometric functions: The graphs of trig functions can be transformed in the same way as other functions. eg:

- $y = \sin 2x$ stretches the graph of $y = \sin x$ by ½ in the x direction – this is equivalent to doubling the period or halving the wavelength.
- $y = 3\sin x$ stretches the graph ×3 in the y-direction, thus trebling the amplitude of the wave.
- $y = \sin x + 1$ moves the graph up 1 in the y-direction, thus shifting its centre of oscillation.
- If all three of the above transformations were performed on the function $f(x) = \sin x$, the resulting graph would be:

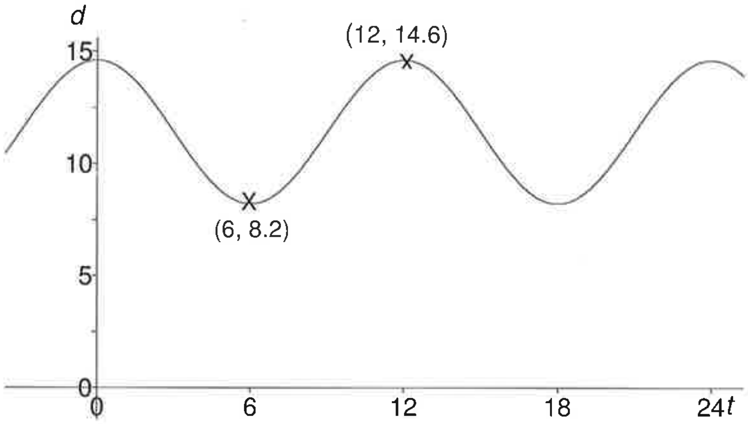


The following is a complete section B question.

A formula for the depth d metres of water in a harbour at time t hours after midnight is

$$d = P + Q\cos(\tfrac{\pi}{6}t),\quad 0 \le t \le 24,$$

where P and Q are positive constants. In the following graph the point (6, 8.2) is a minimum point and the point (12, 14.6) is a maximum point.



a) **Find the values of Q and P.**
b) **Find the first time in the 24 hour period when the depth of water is 10m.**
c) **Use the symmetry of the graph to find the *next* time when the depth is 10m, and hence the time intervals during which the water is less than 10m deep.**

a) The cos curve oscillates about $y = 11.4$ (mean of 8.2 and 14.6) and has an amplitude of 3.2 either side of this value. It follows directly that <u>P = 11.4</u> and <u>Q=3.2</u>. This means that the full equation of the curve is $d = 11.4 + 3.2\cos(\pi t/6)$.

b) To find when the depth is 10 we could plot the curve on the calculator and find when $y = 10$. Alternatively, we could solve $10 = 11.4 + 3.2\cos(\pi t/6)$ and find t.

$10 = 11.4 + 3.2\cos(\pi t/6)$
$-1.4 = 3.2\cos(\pi t/6)$
$-0.4375 = \cos(\pi t/6)$
$2.024 = (\pi t/6)$ *(2.024 is $\cos^{-1}(-0.4375)$ when calculated in radians)*
$t = (6 \times 2.024)/\pi =$ **3.87 hours after midnight** *(Check this looks good on the graph)*

c) The graph is symmetrical about $x = 6$. 3.87 is 2.13 hours *less* than 6, so the next time the depth is 10m will be 2.13 *more* than 6. 6 + 2.13 = **8.13 hours after midnight.**

Looking at the graph, we can see that the water is less than 10m deep between these two times and, using symmetry, it will also be less than 10m deep for 2.13 hours either side of 18. So, the times when the water is less than 10m deep are: **$3.87 \le t \le 8.13$, $15.87 \le t \le 20.13$**

Exact solutions: Particularly on the Paper 1, the solutions are going to be fairly "standard" angles. (You may find it easier to work in degrees, and then convert to radians at the end). It is important to learn the table on page 24, and know how to extend it for angles such as 120°, 270°.

Solve $\sin^2 x + \sin x\cos x = 0$, for $\pi \le x \le 2\pi$

$\sin x(\sin x + \cos x) = 0$
Thus $\sin x = 0$ or $\sin x = -\cos x \Rightarrow \tan x = -1$

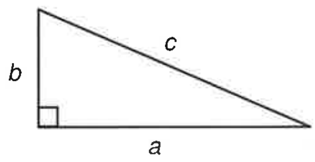
By considering the graphs of $\sin x$ and $\tan x$, and using the table of exact values, we get:

$\sin x = 0 \Rightarrow x = 0,\ \pi,\ 2\pi$

$\tan x = -1 \Rightarrow x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$

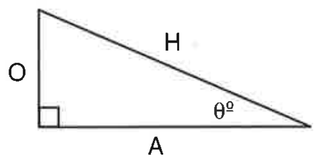
Solution of Triangles

Right-angled triangles: This page is a reminder of how to deal with the sides and angles of a right-angled triangle. The following page deals with non right-angled triangles.

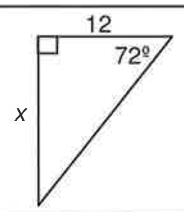


Pythagoras' Theorem: If you know two sides of a right-angled triangle, you can calculate the third using Pythagoras' Theorem. This states that the square of the hypotenuse (the longest side) equals the sum of the squares of the two shorter sides. As applied to the triangle on the left, $c^2 = a^2 + b^2$. You must remember to subtract if you already have the hypotenuse (it's always opposite the right angle) and want to calculate one of the other sides. For example, $b^2 = c^2 - a^2$.

Trigonometry: There is no mystery to sin, cos and tan. They simply represent the ratios of pairs of sides for a triangle with given angles. For example, suppose the smallest angle in the triangle above left is 30°. Whatever the *size* of the triangle, b turns out to be half of c. The ratio of b to c is called the sine (sin for short), so sin30° = 0.5. The ratio of a to c is called the cosine (cos), and b to a is the tangent (tan). If you use the following procedure *in all cases* then every question can be worked out in the same way, and you should always get the right answer.

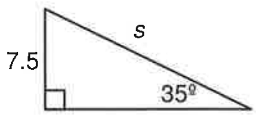


1. Label the three sides of the triangle with H (for hypotenuse, the side opposite the right angle), O (for opposite, the side opposite the angle you are dealing with) and A (for adjacent, the side next to the angle).
2. For the two sides you are dealing with, write down the word sin, cos or tan according to the mnemonic SOH/CAH/TOA.
3. Now write down the angle (which may be unknown) followed by an equals sign.
4. On the right hand side of the equals sign, you will write down a fraction (O over H, A over H or O over A) which will either involve two known sides, or one known and one unknown side.
5. You will now have an equation to solve. The three examples below show how to do this.



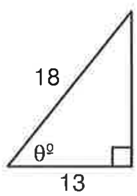
Find x.
x is O, 12 is A, so we use tan.
Write down tan, then the angle, then =, then the fraction O/A.
To solve this equation, just multiply through by 12.

$$\tan 72 = \frac{x}{12}$$
$$12 \times \tan 72 = x$$
$$x = 36.9$$



Find s.
s is H, 7.5 is O, so we use sin.
Write down sin, then the angle, then =, then the fraction O/H *(note that this time the unknown side will be on the bottom of the fraction).*
This time, we must "cross-multiply" to solve the equation.

$$\sin 35 = \frac{7.5}{s}$$
$$s = \frac{7.5}{\sin 35}$$
$$s = 13.1$$



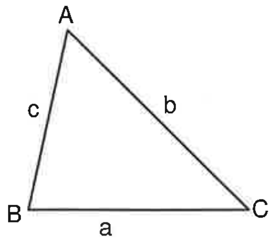
Find the angle θ°.
13 is A, 18 is H, so we use cos.
Write down cos, then the angle, then =, then the fraction A/H.
Calculate the value of the fraction, then use the $\cos^{-1}$ function to find out the angle ($\cos^{-1}$ means "find the angle whose cosine is...)

$$\cos\theta = \frac{13}{18}$$
$$\cos\theta = 0.7222$$
$$\theta = \cos^{-1} 0.7222$$
$$\theta = 43.8°$$

Having worked out 13/18, leave the answer on the display. Then work out the angle using $\cos^{-1}$ANS. This ensures full accuracy.

Sine and Cosine Rules: For triangles which are *not* right-angled we use the sine and cosine rules. The triangle on the right has the conventional notation of small letters for the lengths of sides and capital letters for the angles opposite. To find lengths and angles, use:

- The sine rule if 2 sides and 2 angles are involved
- The cosine rule if 3 sides and 1 angle are involved



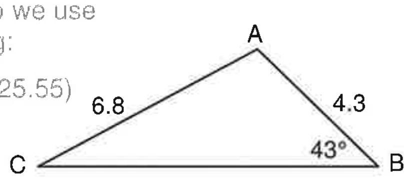
SINE RULE	COSINE RULE
$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$	$a^2 = b^2 + c^2 - 2bc\cos A$ *(for a side)* $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ *(for an angle)*

Don't be put off by the letters. Basically, the sine rule says the ratio of side/sine is the same for each pair of sides and angles. And in the cosine rule, ensure that the side on the LHS of the equation matches the angle on the RHS.

In triangle ABC, angle B = 43°, AC = 6.8 cm and AB = 4.3cm. Find the size of angle A, giving your answer to the nearest degree.

It is essential to draw a rough diagram which will show you how to proceed. We know 2 sides and 1 angle and we want another angle, so we use the sine rule. We can only find angle C at the moment, using:

$\dfrac{4.3}{\sin C} = \dfrac{6.8}{\sin 43}$ which gives C = 25.55°. So A = 180 – (43+25.55)



<u>A = 111.45º = 111° to the nearest degree</u>

A triangle has sides 4, $\sqrt{48}$ and 8. Calculate the size of the angle opposite the side with length $\sqrt{48}$.

Use the cosine rule (in its second form), making sure that the side opposite the angle is also on the left hand side of the formula.

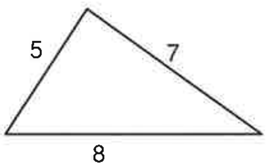
<u>60°</u>

YOU SOLVE

Area of a non-right angled triangle: If you know two sides of a triangle, and the size of the angle between the two sides, then the area of the triangle can be found using:

$$\text{Area} = \frac{1}{2}ab\sin C$$

The diagram shows a triangle with sides 5, 7 and 8. Find the size of the smallest angle and the area of the triangle.

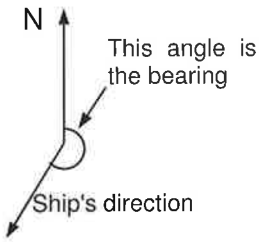


The smallest angle is opposite the smallest side, 5.

$\cos x = \dfrac{7^2 + 8^2 - 5^2}{2 \times 7 \times 8} = 0.786$. So the angle is <u>38.2°</u>

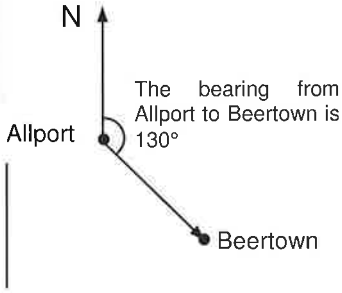
Area = ½ × 7 × 8 × sin38.2° = <u>17.3</u>
(Remember that the angle used in the area formula must be between the two sides used).

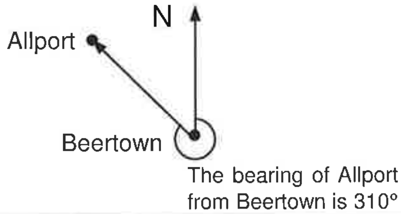
Bearings: One of the practical applications of non-right angled trigonometry is the calculation of distances and angles for moving ships and planes. Their direction of travel is based on compass directions, called *bearings*. A bearing is an angle measured around clockwise from North. Always draw in North lines on your diagrams before marking in bearings.



If a question involves bearings between places, check whether you are dealing with the bearing of A from B or the bearing from A to B, which is the other way round. Use arrows to show in which direction to take the bearing, and put the North line at the *start* of the arrow.

There is always a difference of 180° between bearings in opposite directions.



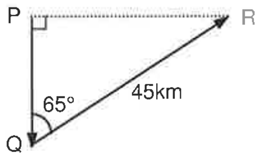


A ship sails from port P and travels due South to port Q. From port Q it sails on a bearing of 065° and travels for 45km to a point R, which is due East of P.

a) i) Draw and label clearly a diagram to show P, Q and R.

ii) Calculate the distance form port P to point R.

In questions like this the diagram is an important tool, so make it large. Angles do not have to be accurate nor lengths drawn to scale, but make them look approximately right.



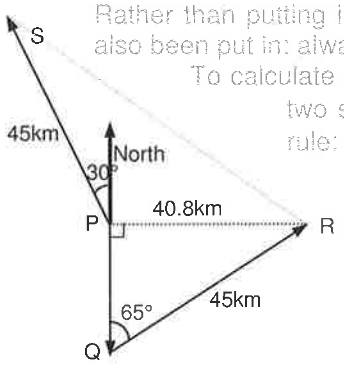
ii) Using SOHCAHTOA (because the triangle is right angled) we can see that $\sin 65 = \frac{PR}{45} \Rightarrow PR = 45\sin 65 = 40.8$

<u>The distance from P to R is 40.8km</u>

A second ship also sails from port P for 45km to a point S, but on a bearing of 330°.

b) Complete your diagram in part (a) to show point S.

c) Calculate the distance from R to S (shown with a grey dotted line) **and the angle PRS.**



Rather than putting in 330°, the more useful 30° has been shown instead. The 40.8 has also been put in: always keep your diagrams up-to-date with new information.

To calculate RS, we use triangle PRS which is not right angled. We already know two sides and one angle (SPR = 30 + 90 = 120°), so we use the cosine rule: $RS^2 = 45^2 + 40.783^2 - 2 \times 45 \times 40.783 \times \cos 120$

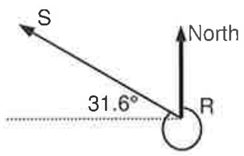
<u>$RS = \sqrt{5523.6} = 74.3$km</u> (Check: RS < RP + PS. Looks OK)

Now we need to calculate angle PRS. We know one angle and two sides so we use the sine rule.

$$\frac{\sin PRS}{45} = \frac{\sin 120}{74.321} \Rightarrow \sin PRS = \frac{45\sin 120}{74.321} = 0.5244$$

<u>So angle PRS = $\sin^{-1}(0.5244) = 31.6°$</u>

d) What is the bearing of S from R?



The diagram shows the arrow representing S from R, and a new North line inserted. The required bearing has also been put in. How big is this angle? From North round to West is 270°, and then we need another 31.6. <u>So, the bearing of S from R = 301.6°</u>

(Note that throughout the question calculations have been performed with numbers to 4 SF accuracy, even if answers are given to 3 SF)