7. Circular functions and trigonometry

Radian measure

You should be able to:

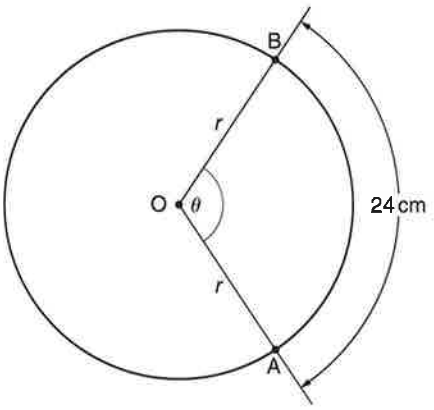
- use the formula $l = \theta r$ to find arc length
- use the formula $A = \frac{1}{2}\theta r^2$ to find the area of a sector.

You should know:

- when an arc that subtends an angle θ at the centre of a circle is equal in length to the radius of the circle, then θ has a measure of 1 radian. For example, if the length of the arc is three times the radius, then the central angle would measure 3 radians
- a sector of a circle is a portion of the circle that is enclosed by two radii and an arc
- the arc length of a sector is a fraction $\left(\frac{\theta}{2\pi}\right)$ of the total circumference of the circle
- similarly the area of a sector is a fraction of the total area of the circle.

Example

The following diagram shows a circle of radius r and centre O. The angle $\hat{AOB} = \theta$.



The length of the arc AB is 24 cm. The area of the sector OAB is $180\,\text{cm}^2$.

Find the values of r and θ.

We are given two pieces of information and are asked to find the values of two unknowns. This suggests that we need to set up a system of equations in r and θ.

We have $l = r\theta = 24$ as our first equation and $A = \frac{1}{2}r^2\theta = 180$ as our second equation. Substituting $r\theta = 24$ into the second equation gives

$\frac{1}{2}r(24) = 180$

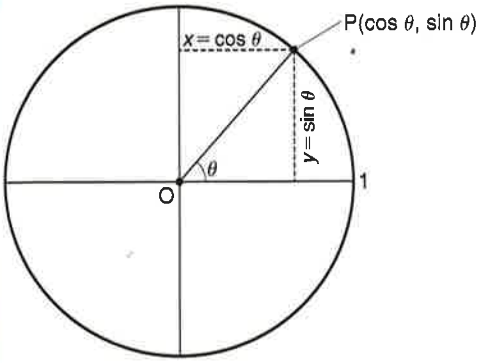
$r = 15\text{ cm}$

Substituting $r = 15$ into the first equation gives $\theta = 1.6$ radians.

Be prepared

- Assume that all angles are given in radians unless otherwise stated.
- Change between radian and degree mode on your calculator.
- Angles measured in radians are often but not always expressed in terms of π; they may also be expressed as decimals.

The unit circle

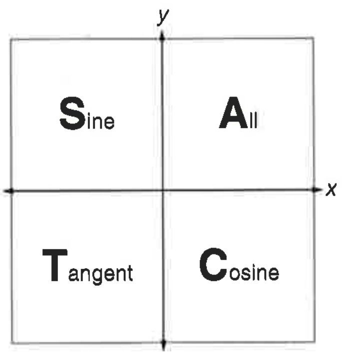


You should be able to:

- define $\sin\theta$ and $\cos\theta$ in terms of the unit circle
- find the value of, say, $\cos\theta$ given the value of $\sin\theta$ without finding the angle θ.

You should know:

- the unit circle has a radius of 1
- a point P that lies on the circumference of the unit circle has coordinates $(\cos\theta, \sin\theta)$
- the symmetry of the circle can be used to demonstrate that the trigonometric functions of any angle are the same in magnitude as the trigonometric functions of the corresponding first quadrant angle; these values will differ in their signs only. For example $\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -0.5$
- the signs of the trigonometric functions depend on the quadrant in which the angle lies. For example, a third quadrant angle has both $\cos\theta$ and $\sin\theta$ negative. Since $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-ve}{-ve} = +ve$, then only tangent is positive in the third quadrant. The following diagram illustrates where the other trigonometric functions are **positive**:



- the gradient of a line passing through the origin is equal to the tangent of the angle that the line makes with the x-axis. Hence, a line through the origin has the equation $y = x\tan\theta$.

Example

Let $p = \sin 40°$ and $q = \cos 110°$. Give your answers to the following in terms of p and/or q.

(a) Write down an expression for

(i) $\sin 140°$

(ii) $\cos 70°$.

You are expected to solve this type of problem without a GDC. We therefore need to make use of the symmetry in the unit circle. The angle 140° is a second quadrant angle and sine is positive in quadrant two. Hence, $\sin 40° = \sin(180 - 40)°$, and $\sin 140° = p$.

Since 110° is a second quadrant angle and cosine is negative in the second quadrant, $\cos 70° = -\cos 110° = -q$.

(b) Find an expression for $\cos 140°$.

This part is best solved using the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$. Making appropriate substitutions and rearranging the identity, we have

$\cos^2 140° = 1 - \sin^2 140°$

$\cos 140° = \pm\sqrt{1 - p^2}$

As 140° is a second quadrant angle, then

$\cos 140° = -\sqrt{1 - p^2}$

You may see that $\cos 140°$ can also be given in terms of q by using the double angle identity, $\cos^2\theta = 2\cos^2\theta - 1$. Try it!

(c) Find an expression for $\tan 140°$.

Following from parts (a) and (b) and using the identity that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we have

$\tan 140° = \frac{\sin 140°}{\cos 140°} = -\frac{p}{\sqrt{1 - p^2}}$

Be prepared

- Consider in which quadrant the angle lies. This will help you determine whether your answer should be positive or negative.

The circular functions

You should be able to:

- sketch the graphs of the circular functions $\sin x$, $\cos x$ and $\tan x$ and composites of these, for example $f(x) = a\sin(b(x + c)) + d$
- state the amplitude, period and horizontal translation of a function of the form $f(x) = a\sin(b(x + c)) + d$
- find the maximum and minimum values of a function of the form $f(x) = a\sin(b(x + c)) + d$
- determine the values of the parameters a, b, c and d from a graph or from information given in a problem
- apply circular functions to problems involving periodic behaviour, such as the ebb and flow of tides, motion of a Ferris wheel, or sound waves.

You should know:

- the period of a function is the number of units required to complete one cycle of the graph
- the functions $y = a\sin(b(x - c)) + d$ and $y = a\cos(b(x - c)) + d$ have amplitude $|a|$, period $\frac{2\pi}{|b|}$, horizontal translation $|c|$ and vertical translation d. If $c < 0$, the graph translates to the left, and if $c > 0$, the graph translates to the right. If $d < 0$, the graph translates downwards, and if $d > 0$, the graph translates upwards
- the parameters a, b, c and d also represent transformations of the graphs of $\sin x$, $\cos x$ and $\tan x$. For example, $g(x) = 2\cos\left(3\left(x - \frac{\pi}{4}\right)\right) + 1$ is the image of the graph of $f(x) = \cos x$ under a vertical stretch factor 2, a horizontal stretch factor $\frac{1}{3}$, a translation of $\frac{\pi}{4}$ units parallel to the x-axis and a translation of 1 unit parallel to the y-axis
- the **amplitude** of a sine or cosine function is half the vertical distance between the highest and lowest points of a graph
- the **period** of a sine or cosine function is best determined by finding the horizontal distance between two successive maximum or minimum points on a graph
- the graph of the tangent function has vertical asymptotes at all integer multiples of $\frac{\pi}{2}$, as the tangent function is not defined at these values
- the function $f(x) = a\tan(b(x + c)) + d$ has a period equal to $\frac{\pi}{|b|}$
- the period of the tangent function can be found by looking at the distance between two consecutive zeros
- the graph of $\cos x$ is simply the graph of $\sin x$ translated $\frac{\pi}{2}$ units to the left—that is, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

Example

The depth, y metres, of seawater in a bay t hours after midnight may be represented by the function

$y = p + q\cos\left(\frac{2\pi}{k}t\right)$, where p, q and k are constants.

The water is at a maximum depth of 14.3 metres at midnight and noon, and is at a minimum depth of 10.3 metres at 06:00 and at 18:00.

(a) Write down the value of p.

The value of p represents the vertical translation of the graph of y = cos t. This translation can be determined by finding the mid point of the maximum and minimum values. That is,

$$p = \frac{14.3 + 10.3}{2} = 12.3$$

(b) Write down the value of q.

The value of q represents the amplitude of the graph of this function. The amplitude can be determined by finding half the distance between the maximum and minimum values. That is,

$$q = \frac{14.3 - 10.3}{2} = 2$$

(c) Write down the value of k.

Using the fact that the period is $\frac{2\pi}{|b|}$ and our function has $b = \frac{2\pi}{k}$, the period is equal to k. We need only to find the period from the given information. As the maximum depth occurs at midnight and noon, the time between consecutive maximum values is 12 hours. Hence, the period (and k) is 12.

The circular functions (continued)

Be prepared

- A function of the form $y = a\sin(bx + c) + d$ has a horizontal translation of $\frac{c}{b}$ units to the left.
- Use the correct **mode** when graphing functions on your GDC. This will normally be radian mode unless otherwise stated.
- Entering the circular functions into your GDC can be a tricky exercise. Pay close attention to putting the brackets in the correct place.
- When finding the maximum or minimum value of a sine or cosine function, remember to consider the vertical translation as well as the amplitude. That is, the maximum value of the function $g(x) = 2\cos\left(3\left(x - \frac{\pi}{4}\right)\right) + 1$ is $1 + 2 = 3$ and the minimum value is $1 - 2 = -1$.

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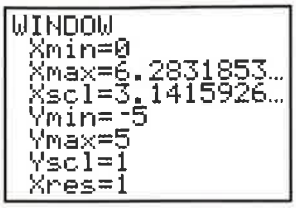

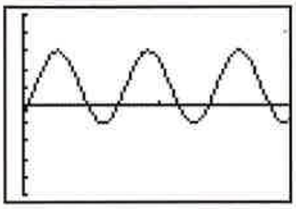

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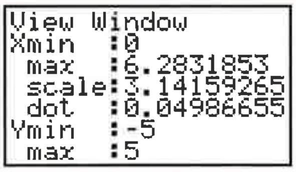

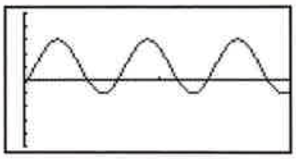

Identities and equations

You should be able to:

- use the double angle formulae (identities) for sine and cosine to simplify expressions and solve equations
- solve trigonometric equations in a given finite interval both analytically and graphically
- use the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ to express sine in terms of cosine or cosine in terms of sine and simplify expressions
- determine the number of solutions that an equation has in a given interval.

You should know:

- an identity is different from an equation in that an identity is true for all values of a variable
- a trigonometric equation is an equation that contains trigonometric functions
- the simplest form of a trigonometric equation is written, for example, as $\sin\theta = k$. The solution(s) to this equation are the values of θ for which the sine is k. For example, in the first quadrant, if $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$
- trigonometric identities can be used to write trigonometric equations in their simplest form
- some trigonometric equations can be expressed in quadratic forms that can be factorized and then solved.

Example

Consider the equation $3\cos 2x + \sin x = 1$.

(a) Write this equation in the form $f(x) = 0$, where $f(x) = p\sin^2 x + q\sin x + r$, and $p, q, r \in \mathbb{Z}$.

The question is asking us to express $f(x)$ completely in terms of $\sin x$. To do this, we make use of the double angle identity $\cos 2x = 1 - 2\sin^2 x$. Therefore,

$3(1 - 2\sin^2 x) + \sin x = 1$

Simplifying and making the coefficient of $\sin^2 x$ positive, we have

$3 - 6\sin^2 x + \sin x = 1$

$6\sin^2 x - \sin x - 2 = 0$

(b) Factorize $f(x)$.

If we let $\sin x = k$, for example, we can write a quadratic expression in terms of k as $6k^2 - k - 2$. This factorizes into $(3k - 2)(2k + 1)$. Substituting $\sin x$ back in for k, we have $(3\sin x - 2)(2\sin x + 1)$

(c) Write down the number of solutions of $f(x) = 0$, for $0 \le x \le 2\pi$.

*From (b) and using an **analytical approach**, $\sin x = \frac{2}{3}$ or $\sin x = -\frac{1}{2}$. The solutions **to these two equations** would result in an angle in all four quadrants. Hence, there are four solutions to $f(x) = 0$.*

Using a graphical approach, we see that there are, again, four solutions. Notice that the window has been conveniently set to match the given domain of $0 \le x \le 2\pi$.

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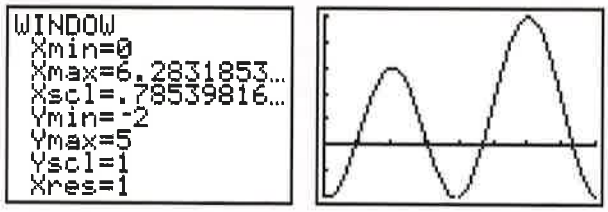

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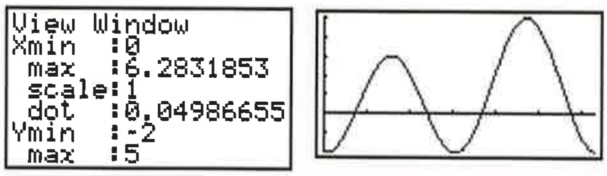


Be prepared

- It is often helpful to write a trigonometric equation in terms of only one trigonometric function before attempting to solve the equation analytically.
- Write down only those solutions to an equation that belong in the given interval. Extra solutions are often penalized.
- Use the "intersect" feature of your GDC to find solutions to trigonometric equations. Set your window to match the given domain to avoid additional solutions.

Solution of triangles

You should be able to:

- define the sine, cosine and tangent ratios for right-angled triangles
- use the sine and cosine rules to find unknown sides and angles in non-right-angled triangles
- use the sine and cosine rules to solve problems set in real-life contexts such as navigation
- recognize the ambiguous case of the sine rule
- calculate the area of non-right-angled triangles.

You should know:

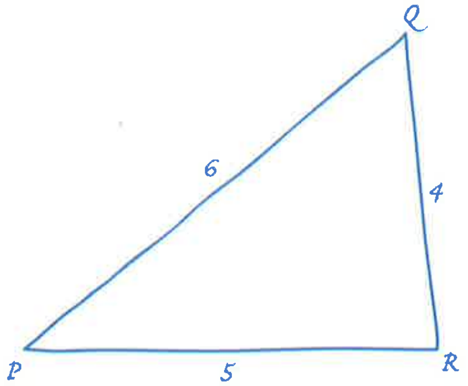
- the sine rule can only be used on a triangle for which an angle and the side **opposite** that angle are known
- an ambiguous case may arise whenever the sine rule is used to find a missing angle. This is due to the fact that there are two angles, θ and its supplement, that have the same sine, that is, $\sin\theta = \sin(180° - \theta)$
- the cosine rule can be used to find a missing side in a triangle if the lengths of two sides and the size of the angle between them are known
- the cosine rule can also be used to find a missing angle in a triangle if the lengths of all three sides of the triangle are known
- the area of a non-right-angled triangle can be found if you know the lengths of two sides and the size of the angle between them.

Example

In the triangle PQR, PR = 5 cm, QR = 4 cm and PQ = 6 cm.

(a) Calculate the size of $P\hat{Q}R$.

Start by sketching triangle PQR to show the given information.



As the lengths of three sides are given, we can use the cosine rule to find the angle at Q.

$$\cos Q = \frac{6^2 + 4^2 - 5^2}{2(6)(4)} = 0.5625$$

$P\hat{Q}R \approx 0.973$ radians correct to three significant figures.

(b) Calculate the area of triangle PQR.

Using $P\hat{Q}R$ and the formula for the area of a triangle, we have:

$$A = \frac{1}{2}(6)(4)\sin 0.973\ldots$$

$A \approx 9.92$ correct to three significant figures.

Be prepared

- Sketch diagrams before applying any rules.
- Solve triangles given in a three-dimensional figure.
- Consider the ambiguous case whenever the sine rule is applied to find a missing angle. You may see certain words, phrases or diagrams that suggest its occurrence. For example, two angles with the same sine, two triangles, supplementary angles, and so on.