Sequences and Series

Topics

- Sequences.
- Limits of sequences.
- Summation notation.
- Series.
- Geometric series.

Definitions

- An **infinite sequence** is a function whose domain is the set of positive integers. The function values $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the terms of the sequence.

- The **Fibonacci sequence** is defined by $a_0 = 1,\ a_1 = 1,\ a_k = a_{k-2} + a_{k-1},\ k > 1$.

- **Factorial** notation: $n! = 1(2)(3)\cdots(n-1)(n)$. $0! = 1$.

- The **sum** of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots a_n$. The index of summation is i, n is the upper limit of summation, and 1 is the lower limit.

- The sum of the terms of an infinite geometric sequence is called an **infinite geometric series** or simply a **geometric series**.

Theorems

- Sum of a finite geometric series: $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$.

- Sum of an infinite geometric series: $\sum_{i=0}^{\infty} r^i = \frac{1}{1-r},\ |r| < 1$.

  (If $|r| \geq 1$, the infinite series does not have a sum.)

Example 1: Writing the Terms of a Sequence

Write out the first 3 terms of the following sequences, beginning with $n = 1$.

1. $a_n = 3 + (-1)^n$

2. $a_n = \left(\frac{1}{2}\right)^n$

Solution

1. $a_1 = 3 + (-1)^1 = 2,\ a_2 = 3 + (-1)^2 = 4,\ a_3 = 3 + (-1)^3 = 2$

2. $a_1 = \left(\frac{1}{2}\right)^1 = \frac{1}{2},\ a_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4},\ a_3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

A key issue in the theory of sequences is that of limits. Informally, a sequence has a limit L if the terms of the sequence get arbitrarily close to L as n tends to infinity. In Example 1 above, the second sequence has limit 0, whereas the first sequence does not have a limit.

Example 2: The Fibonacci Sequence

The Fibonacci sequence is described by a recursive formula. Write out the first 8 terms of the Fibonacci sequence given that $a_1 = 1,\ a_2 = 1,$ and $a_k = a_{k-2} + a_{k-1}$ for $k = 2,\ 3,\ 4,\ \ldots$.

Solution

We know that the first 2 terms are 1 and 1. The third term is the sum of the first 2 terms, $a_3 = a_1 + a_2 = 1 + 1 = 2$. The fourth term is $a_4 = a_2 + a_3 = 1 + 2 = 3$. Continuing in this manner, you obtain $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$.

Example 3: Summation Notation

Summation notation is used to describe the sum of a finite number of terms of a sequence.

1. $\sum_{i=1}^{4} 3i = 3(1) + 3(2) + 3(3) + 3(4) = 3 + 6 + 9 + 12 = 30$

2. $\sum_{n=0}^{8} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{8!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{40{,}320} \approx 2.71828$

Do you recognize the sum in this second example? If you were to add more and more terms, the sum would approach the number e.

Example 4: A Geometric Series

The theory of infinite series is major topic in calculus courses. One of the most familiar infinite series is the geometric series: $\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-r} = \frac{1}{1-\left(\frac{1}{2}\right)} = 2.$

Study Tips

- It is often convenient to begin subscripting a sequence with n = 0 instead of $n = 1$ instead of In fact, a sequence can begin with any value of n. Furthermore, any convenient letter can be used as the index of summation.
- Graphing utilities have a built-in feature for calculating factorials. Try calculating $20!$ with your graphing utility. The answer will have 19 digits!
- The formula for a sequence is not unique. For example, the following 3 rules describe the same sequence of odd numbers.

  1. $a_n = 2n-1,\ n=1,\ 2,\ 3,\ ...$

  2. $b_n = 2n+1,\ n=0,\ 1,\ 2,\ ...$

  3. $c_n = 2n-3,\ n=2,\ 3,\ 4,\ ...$

Pitfall

- Real numbers can have multiple representations. For example, the number 1 is equal to the infinite decimal expansion $0.99999...$.

Problems

1. Write the first 5 terms of the following sequences. Assume n begins with 1.

a. $a_n = 2n+5$

b. $a_n = \dfrac{n+1}{n}$

2. Write an expression for the apparent n^{th} term of the following sequences.

a. $1,\ 4,\ 7,\ 10,\ 13,\ ...$

b. $0,\ 3,\ 8,\ 15,\ 24,\ ...$

3. Simplify the following factorial expressions.

a. $\dfrac{12!}{4!\,8!}$

b. $\dfrac{(2n-1)!}{(2n+1)!}$

4. Find the following sums.

a. $\sum_{i=1}^{5}(2i+1)$

b. $\sum_{i=0}^{4} i^2$

5. Find the sum of the following infinite series.

a. $\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^n$

b. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^n$

c. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$

Answers

1. **a.** $a_1 = 2(1)+5 = 7,\ a_2 = 2(2)+5 = 9,\ a_3 = 2(3)+5 = 11$
$a_4 = 2(4)+5 = 13,\ a_5 = 2(5)+5 = 15$

b. $a_1 = \frac{1+1}{1} = 2,\ a_2 = \frac{2+1}{2} = \frac{3}{2},\ a_3 = \frac{4}{3},\ a_4 = \frac{5}{4},\ a_5 = \frac{6}{5}$

2. **a.** $a_n = 3n-2$ (Beginning with $n=1$. There are many possible answers.)

b. $a_n = n^2 - 1$ (Beginning with $n=1$. There are many possible answers.)

3. **a.** $\frac{12!}{4!\,8!} = \frac{12(11)(10)(9)8!}{4!8!} = \frac{12(11)(10)(9)}{4!} = \frac{11{,}880}{24} = 495$

b. $\frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)}$

4. **a.** $\sum_{i=1}^{5}(2i+1) = (2+1)+(4+1)+(6+1)+(8+1)+(10+1) = 35$

b. $\sum_{i=0}^{4} i^2 = 0^2+1^2+2^2+3^2+4^2 = 30$

5. **a.** $\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^n = 10\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n = 10\left(\frac{1}{1-4/5}\right) = 50$

b. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^n = 5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 5\frac{1}{1-\left(-1/2\right)} = \frac{10}{3}$

c. The series does not have a finite sum because $\frac{7}{3} > 1$.